ANSWERS

FOR THE
IB DIPLOMA
PROGRAMME

Mathematics

ANALYSIS AND APPROACHES HL

EXAM PRACTICE WORKBOOK

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Boost
HODDER EDUCATION

Answers to Practice Questions

1 Number and algebra

1 Just plug the numbers into your calculator. The answer is 4×10^{80}

2 $$\begin{aligned} 3 \times 10^{97} - 4 \times 10^{96} &= 10^{96}(3 \times 10^1 - 4) \\ &= 10^{96}(30-4) \\ &= 26 \times 10^{96} \\ &= 2.6 \times 10 \times 10^{96} \\ &= 2.6 \times 10^{97} \end{aligned}$$

3 We can write this expression as

$$\begin{aligned} \left(\frac{6}{8}\right) \times \left(\frac{10^{30}}{10^{-12}}\right) &= 0.75 \times 10^{30-(-12)} \\ &= 7.5 \times 10^{-1} \times 10^{42} \\ &= 7.5 \times 10^{41} \end{aligned}$$

4 The first term is 20. The common difference is -3. Therefore

$$\begin{aligned} u_{25} &= 20 + (-3) \times (25-1) \\ &= 20 - 3 \times 24 \\ &= -52 \end{aligned}$$

Tip: If this had been a calculator question, you could have just used $u_1 = 20, u_{n+1} = u_n - 3$ in your calculator sequence function.

5 $u_n = 1602 = 21 + 17(n-1)$

$1581 = 17(n-1)$

$93 = n - 1$

$94 = n$

6 $u_4 = 10 = u_1 + 3d$

$u_{10} = 34 = u_1 + 9d$

Subtracting gives;

$24 = 6d$

$4 = d$

Substituting into the first equation:

$10 = u_1 + 3 \times 4$

$u_1 = -2$

Therefore

$$\begin{aligned} u_{20} &= -2 + 19 \times 4 \\ &= 74 \end{aligned}$$

Tip: If this had been a calculator question, you could have solved the simultaneous equations using your GDC.

7 $u_1 = 13, d = -3$ so

$$S_{30} = \frac{30}{2}\big(2 \times 13 - 3 \times (30 - 1)\big) = -915$$

8 $S_{20} = \frac{20}{2}(4 + 130) = 1340$

9 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 16 + 21 + 26 \ldots$

So the first term is 16 and the common difference is 5.

10 Your GDC should have a sum function, which can be used for this. The answer will be 26350, but you should still write down the first term and common difference found in question 9 as part of your working.

11 This is an arithmetic sequence with first term 500 and common difference 100. The question is asking for S_{28}.

$$S_{28} = \frac{28}{2}(2 \times 500 + 100 \times 27) = 51800 \text{ m}$$

Tip: The hardest part of this question is realising that it is looking for the sum of the sequence, rather than just how far Ahmed ran on the 28th day.

12 2.4% of \$300 is \$7.20. This is the common difference. After one year, there is \$307.20 in the account, so this is the 'first term'.

$u_{10} = 307.20 + 9 \times 7.20 = \$372$

Tip: The hardest part of this question is being careful with what 'after 10 years' means – it is very easy to be out by one year.

13 a The differences in velocity are 1.1, 0.8 and 0.8. Their average is 0.9. When $t = 0.5$ we are looking for the sixth term of the sequence which would be

$u_6 = 0 + 0.9 \times 5 = 4.5 \text{ m s}^{-1}$

b There are many criticisms which could be made about this model – for example:

- There is too little data for it to be reliable.
- There is no theoretical reason given for it being an arithmetic sequence.
- The ball will eventually hit the ground.
- The model predicts that the ball's velocity grows without limit.
- There seems to be a pattern with smaller differences later on.

14 The first term is 32 and the common ratio is $-\frac{1}{2}$.

$$u_{12} = 32 \times \left(-\frac{1}{2}\right)^9 = -\frac{1}{16}$$

15 The first term is 1 and the common ratio is 2 so

$u_n = 1 \times 2^{n-1}$

If $u_n = 4096$ then

$4096 = 2^{n-1}$

There are four ways you should be able to solve this:

- On a non-calculator paper you might be expected to figure out that $4096 = 2^{12}$
- You can take logs of both sides to get $\ln 4096 = (n-1)\ln 2$ and solve for n.
- You can graph $y = 2^{x-1}$ and intersect it with $y = 4096$
- You can create a table showing the sequence and determine which row 4096 is in.

Whichever way, the answer is 13.

16 $u_3 = u_1 r^2 = 16$

$u_7 = u_1 r^6 = 256$

Dividing the two equations:

$$\frac{u_1 r^6}{u_1 r^2} = \frac{256}{16}$$

$r^4 = 16$

$r = \pm 2$

$u_1 = \frac{16}{r^2} = 4$ for both possible values of r.

17 The first term is 162, the common ratio is $\frac{1}{3}$

$$S_8 = \frac{162\left(1 - \frac{1}{3}^8\right)}{1 - \frac{1}{3}} = \frac{6560}{27} \approx 243$$

Tip: You can always use either sum formula, but generally if r is between 0 and 1 the second formula avoids negative numbers.

18 The easiest way to deal with sigma notation is to write out the first few terms by substituting in $r = 1, r = 2, r = 3$, etc.:

$S_n = 10 + 50 + 250 \ldots$

So the first term is 10 and the common ratio is 5.

19 Your GDC should have a sum function, which can be used for this. The answer will be 24414060, but you should still write down the first term and common ratio found in question 18 as part of your working.

20 a This is a geometric sequence with first term 50,000 and common ratio 1.2. ‘After 12 days’ corresponds to the 13th term of the sequence so:

$u_{13} = 50{,}000 \times 1.2^{12} = 445805$

b The model suggests that the number of bacteria can grow without limit.

21 You could use the formula:

$$FV = 2000 \times \left(1 + \frac{4}{100 \times 12}\right)^{12\times 10} = £2981.67$$

However, the general expectation is that you would use the TVM package on your calculator for this type of question. Make sure you can get the same answer using your package as different calculators have slightly different syntaxes.

22 Using the TVM package, $i = 5.10\%$

23 Unless stated otherwise, you should assume that the compound interest is paid annually. Using $FV = 200$ and $PV = 100$, the TVM package suggests that 33.35 years are required. There 33 years would be insufficient so 34 complete years are required.

24 12% annual depreciation is modelled as compound interest with an interest rate of –12%. Using the TVM package the value is $24000 to three significant figures.

25 When adjusting for inflation, the 'real' interest rate is $3.2 - 2.4 = 0.8\%$. Using this value in the TVM package we find a final value of $\$2081$

26 This expression is $2^{-2\times -2} = 2^4 = 16$

27 $(2x)^3 = 2^3 \times x^3 = 8x^3$

28 $10^x = \frac{5}{4}$

This is equivalent to $x = \log_{10}\left(\frac{5}{4}\right)$

29 The given statement is equivalent to

$2x - 6 = \ln 5$

So

$2x = \ln 5 + 6$

$x = \frac{1}{2}\ln 5 + 3$

Tip: The answer could be written in several different ways – for example, $\ln(\sqrt{5}\, e^3)$. Generally speaking any correct and reasonably simplified answer would be acceptable.

30 Using appropriate calculator functions:

$\ln 10 + \log_{10} e \approx 2.30 + 0.434 \approx 2.74$

31 LHS $= \frac{m+1}{(m-1)(m+1)} + \frac{m-1}{(m+1)(m-1)}$

$= \frac{m+1+m-1}{m^2 - m + m - 1}$

$= \frac{2m}{m^2 - 1}$

$=$ RHS

32 a $x - ax = b$

$x(1-a) = b$

$x = \frac{b}{1-a}$

b Comparing coefficients of x:

$1 = a$

Comparing constant terms:

$0 = b$

Tip: If you are not familiar with comparing coefficients, you can also substitute in $x = 0$ and $x = 1$ to set up some simultaneous equations to get the same results.

33 $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

34 If $x = \log_4 32$ this is equivalent to $4^x = 32$. There are many ways to proceed, but we could write everything in terms of powers of 2:

$(2^2)^x = 2^5$

$2^{2x} = 2^5$

Therefore, $2x = 5$ so $x = 2.5$

35 $x = \log_2(2 \times 5)$
$= \log_2 2 + \log_2 5$
$= 1 + y$

36 $\log_5 12 = \frac{\ln 12}{\ln 5}$

37 $\ln 5^{x-1} = \ln(4 \times 3^{2x})$

$$(x-1)\ln 5 = \ln 4 + \ln 3^{2x}$$
$$x\ln 5 - \ln 5 = \ln 4 + 2x\ln 3$$
$$x\ln 5 - 2x\ln 3 = \ln 4 + \ln 5$$
$$x(\ln 5 - 2\ln 3) = \ln 4 + \ln 5$$
$$x = \frac{\ln 4 + \ln 5}{\ln 5 - 2\ln 3}$$
$$= \frac{\ln 20}{\ln 5 - \ln 9}$$
$$= \frac{\ln 20}{\ln\left(\frac{5}{9}\right)}$$

Tip: In the calculator paper, if an exact form is not required then this type of equation is best solved using an equation solver or a graphical method.

38 The first term is 2, the common ratio is $\frac{1}{3}$, so

$$S_\infty = \frac{2}{1-\frac{1}{3}} = 3$$

39 The common ratio is $2x$. This will converge if $|2x|<1$ which is when $|x| < \frac{1}{2}$. You could also write this as $-\frac{1}{2} < x < \frac{1}{2}$.

40 $(2+x)^4 = 2^4 + {}^4C_1 \times 2^3 \times x + {}^4C_2 \times 2^2 \times x^2 + {}^4C_3 \times 2^1 \times x^3 + x^4$
$= 16 + 4 \times 8 \times x + 6 \times 4 \times x^2 + 4 \times 2 \times x^3 + x^4$
$= 16 + 32x + 24x^2 + 8x^3 + x^4$

41 Use $x = 0.01$, then

$$\begin{aligned}(2+0.01)^4 &= 16 + 32 \times 0.01 + 24 \times 0.0001 \ldots \\ &\approx 16 + 0.32 \\ &= 16.32\end{aligned}$$

42 You could find the full expansion, but that would waste time (especially if the brackets were raised to a larger exponent). The general term is

$$^4C_r(2x)^r\left(-\frac{1}{x^3}\right)^{4-r} = {}^4C_r 2^r(-1)^{4-r}x^r x^{-3(4-r)}$$

$$= {}^4C_r 2^r(-1)^{4-r}x^{4r-12}$$

For this to be a constant, we need $4r - 12 = 0$, so $r = 3$

The term is then $^4C_3 2^3(-1)^{4-3} = -32$

43 $^8C_3 = \frac{8!}{3!5!}$

$$= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{(1 \times 2 \times 3) \times (1 \times 2 \times 3 \times 4 \times 5)}$$

$$= \frac{6 \times 7 \times 8}{1 \times 2 \times 3}$$

$$= \frac{6 \times 7 \times 8}{6}$$

$$= 7 \times 8 = 56$$

Tip: You could also find this by looking at the appropriate number in Pascal's triangle.

44 There are 7 letters so there are $7! = 5040$ possible arrangements.

45 The order in which people are picked does not change the committee, so this is $^{10}C_4 = 210$

46 The order here matters. For example, 134 is different from 413. There are $^5P_3 = 60$ ways.

47 There are two possibilities – either it ends with a 2 or a 4. For each of these there are $^4P_3 = 24$ ways to select the first three digits, making 48 ways in total.

48 $(1-2x)^{\frac{1}{2}} = 1 + \frac{1}{2} \times (-2x) + \frac{1}{2} \times \left(-\frac{1}{2}\right) \times \frac{(-2x)^2}{2!} + \cdots$

$$= 1 - x - \frac{x^2}{2} + \cdots$$

49 $\frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right)$

$$= \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$

$$= \frac{1}{2}(1 + (-1) \times \left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \cdots$$

$$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} + \cdots$$

50 The expansion is valid if $\left|\frac{x}{2}\right| < 1$ so $|x| < 2$

51 $\frac{x+4}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$

$x + 4 \equiv A(x + 1) + B(x - 2)$

If $x = 2 : 6 = 3A$

$A = 2$

If $x = -1$

$3 = -3B$

$B = -1$

Therefore

$$\frac{x+4}{(x-2)(x+1)} \equiv \frac{2}{x-2} - \frac{1}{x+1}$$

52 $wz = (1 - 2i)(2 + i) = 2 + i - 4i - 2(i^2) = 4 - 3i$

Therefore

$2z + wz = (4 + 2i) + (4 - 3i) = 8 - i$

53 $\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2-b^2i^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$

So, the real part is $\frac{a}{a^2+b^2}$

54 Let $z = a + ib$

$(a + ib) + 2(a - ib) = 4 + 6i$

$3a - ib = 4 + 6i$

Comparing real and imaginary parts:

$3a = 4$ so $a = \frac{4}{3}$

$-b = 6$ so $b = -6$

So, $z = \frac{4}{3} - 6i$

55 $|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$

$\tan(\arg z) = \frac{-\sqrt{3}}{1}$

$\arg z = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$

Considering the position of z on the complex plane, $\arg z = -\frac{\pi}{3}$

($\frac{5\pi}{3}$ would also be an acceptable answer)

56 $|z| = \sqrt{2^2 + 2^2} = \sqrt{8}$

$\arg z = \frac{\pi}{4}$

So, $z = \sqrt{8}\,\text{cis}\left(\frac{\pi}{4}\right)$

57 Write $z = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$ in cartesian form.

$z = 4\cos\left(\frac{\pi}{6}\right) + 4i\sin\left(\frac{\pi}{6}\right)$

$= \frac{4\sqrt{3}}{2} + \frac{4i}{2}$

$= 2\sqrt{3} + 2i$

58 First, write each part of the sum in cartesian form.

$e^{\frac{i\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$e^{\frac{i\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$\operatorname{Re}\left(e^{\frac{i\pi}{3}} + 2e^{\frac{i\pi}{6}}\right) = \frac{1}{2} + \sqrt{3}$

59 $2\operatorname{cis}\left(\frac{\pi}{3}\right) \times 3\operatorname{cis}\left(\frac{2\pi}{3}\right) = 6\operatorname{cis}\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) = 6\operatorname{cis}\pi = -6$

60 $(2+i)e^{\frac{i\pi}{3}} = (2+i)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 1 - \frac{\sqrt{3}}{2} + i\left(\frac{1}{2} + \sqrt{3}\right)$

61 Since the cubic has real coefficients, the roots occur in conjugate pairs, so $x = 1 - i$ is also a solution. This means that $(x - 1 - i)$ and $(x - 1 + i)$ are both factors, so together they form the factor:

$(x - 1 - i)(x - 1 + i) = (x - 1)^2 - i^2 = x^2 - 2x + 2$

Dividing into the polynomial, $x^3 - 4x^2 + 6x - 4 = (x^2 - 2x + 2)(x - 2)$

So, the final root is $x = 2$

62 $z = 2^4\operatorname{cis}\left(4 \times \frac{\pi}{4}\right) = 16\operatorname{cis}\pi = -16$

63 From DeMoivre's theorem:

$\cos 3x + i\sin 3x = (\cos x + i\sin x)^3$

Using the binomial expansion:

$= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x$

Comparing real parts:

$\cos 3x = \cos^3 x - 3\cos x\sin^2 x$

Using the identity $\sin^2 x = 1 - \cos^2 x$:

$\cos 3x = \cos^3 x - 3\cos x\,(1 - \cos^2 x)$

$= 4\cos^3 x - 3\cos x$

64 First write $8i$ in polar form, finding three different forms since we are effectively cube rooting each side:

$$8i = 8\,\text{cis}\,\frac{\pi}{2} \text{ or } 8\,\text{cis}\,\frac{5\pi}{2} \text{ or } 8\,\text{cis}\,\frac{9\pi}{2}$$

$$z = (8i)^{\frac{1}{3}} = \left(8\,\text{cis}\,\frac{\pi}{2}\right)^{1/3} \text{ or } \left(8\,\text{cis}\,\frac{5\pi}{2}\right)^{1/3} \text{ or } \left(8\,\text{cis}\,\frac{9\pi}{2}\right)^{1/3}$$

$$= 8^{\frac{1}{3}}\,\text{cis}\left(\frac{\pi}{2}\times\frac{1}{3}\right), \quad 8^{\frac{1}{3}}\,\text{cis}\left(\frac{5\pi}{2}\times\frac{1}{3}\right), \quad 8^{\frac{1}{3}}\,\text{cis}\left(\frac{9\pi}{2}\times\frac{1}{3}\right)$$

$$= 2\,\text{cis}\,\frac{\pi}{6}, \quad 2\,\text{cis}\left(\frac{5\pi}{6}\right), 2\,\text{cis}\left(\frac{3\pi}{2}\right)$$

Tip: You could write this as $z = \sqrt{3} + i,\ z = -\sqrt{3} + i,\ z = -2i.$

65 When $n = 1$:

LHS $= 2 \times 1 - 1 = 1$

RHS $= 1^2 = 1$

So, the statement is true when $n = 1$

Assume that the statement is true when $n = k$

$$\sum_{r=1}^{k} 2r - 1 = k^2$$

Then:

$$\sum_{r=1}^{k+1} 2r - 1 = \left(\sum_{r=1}^{k} 2r - 1\right) + 2(k+1) - 1$$

$$= k^2 + 2k + 2 - 1$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

So, if $n = k$ is true then $n = k + 1$ is also true. Since $n = 1$ is true, the statement is true for all positive integer n.

66 When $n = 1$:

$7^1 - 2^1 = 5$

So, the statement is true when $n = 1$

Assume that the statement is true when $n = k$

$7^k - 2^k = 5A$

where A is an integer.

Then:

$7^{k+1} - 2^{k+1} = 7 \times 7^k - 2 \times 2^k$

$= 7 \times (5A + 2^k) - 2 \times 2^k$

$= 35A + 7 \times 2^k - 2 \times 2^k$

$= 35A + 5 \times 2^k$

$= 5(7A + 2^k)$

which is divisible by 5.

So, if $n = k$ is true then $n = k + 1$ is also true. Since $n = 1$ is true, the statement is true for all positive integer n.

67 When $n = 1$:

$\text{LHS} = \cos\theta + i\sin\theta = \text{RHS}$

So, the statement is true when $n = 1$

Assume that the statement is true when $n = k$

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Then:

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i\,(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

Using the compound angle theorem:

$= \cos(k+1)\theta + i\sin(k+1)\theta$

So, if $n = k$ is true then $n = k + 1$ is also true. Since $n = 1$ is true, the statement is true for all positive integer n.

68 Suppose that $\frac{p}{q} = \sqrt[3]{5}$ where p and q have no common factors above 1.

Then $\frac{p^3}{q^3} = 5$

$p^3 = 5q^3$

So, p^3 is divisible by 5, therefore p is divisible by 5 so let $p = 5r$

$\frac{125r^3}{q^3} = 5$ so $q^3 = 25r^3$ therefore q is divisible by 5. This contradicts the fact that p and q have no common factor above 1, so $\sqrt[3]{5}$ cannot be written as a fraction, so is irrational.

69 When $n = 41$ all three terms are divisible by 41 so the sum is divisible by 41, therefore this is not prime.

70 Using the GDC, $x = 1, y = 2, z = 3$

71 Adding the first two equations:

$2x + 4y + 6z = 2$

Dividing by 2:

$x + 2y + 3z = 1$

This contradicts the third equation, so the system is inconsistent.

72 Eliminating z, $[1] + [2]$:

$2x - y = 7 \quad [4]$

$[1] + [3]$:

$4x - 2y = 14$

$2x - y = 7 \quad [5]$

Since equations [4] and [5] are identical, there are infinite solutions.

Setting $x = \lambda$ in equation [4]:

$y = 2\lambda - 7$

Substituting into 1:

$\lambda + (2\lambda - 7) - z = 2$

$z = 3\lambda - 9$

So, the general solution is $x = \lambda, \;\; y = 2\lambda - 7, \;\; z = 3\lambda - 9$

2 Functions

1 Rearrange into the form $y = mx + c$:

$$\begin{aligned} 3x - 4y - 5 &= 0 \\ 4y &= 3x - 5 \\ y &= \frac{3}{4}x - \frac{5}{4} \end{aligned}$$

So, $m = \frac{3}{4}, c = -\frac{5}{4}$

2 Use $y - y_1 = m(x - x_1)$:

$$\begin{aligned} y + 4 &= -3(x - 2) \\ y + 4 &= -3x + 6 \\ y &= -3x + 2 \end{aligned}$$

3 Find the gradient using $m = \frac{y_2 - y_1}{x_2 - x_1}$:

$$m = \frac{1+5}{9+3} = \frac{1}{2}$$

Use $y - y_1 = m(x - x_1)$:

$$\begin{aligned} y - 1 &= \frac{1}{2}(x - 9) \\ 2y - 2 &= x - 9 \\ x - 2y - 7 &= 0 \end{aligned}$$

4 Gradient of parallel line is $m = 2$

$y - 4 = 2(x - 1)$

$y = 2x + 2$

5 Gradient of perpendicular line is $m = -\frac{1}{-\frac{1}{4}} = 4$

$y - 3 = 4(x + 2)$

$y = 4x + 11$

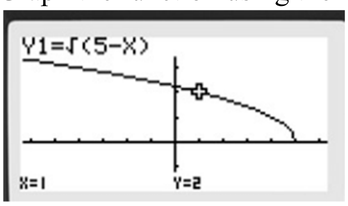
6 Substitute $x = -2$ into the function:

$$\begin{aligned} \mathrm{f}(-2) &= 3(-2)^2 - 4 \\ &= 8 \end{aligned}$$

7 $2x - 1 > 0$

$x > \frac{1}{2}$

8 Graph the function using the GDC:

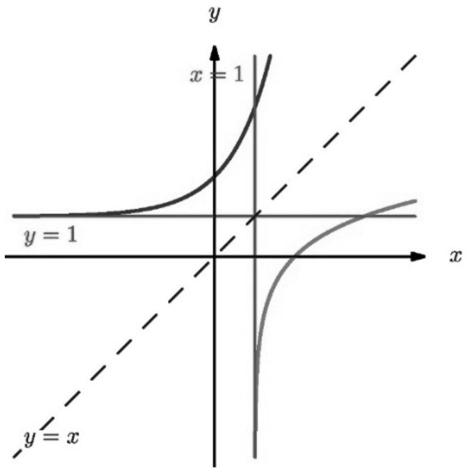


$\mathrm{f}(1) = 2$, so range is $\mathrm{f}(x) \geq 2$

9 To find $f^{-1}(-8)$, solve $f(x) = -8$:

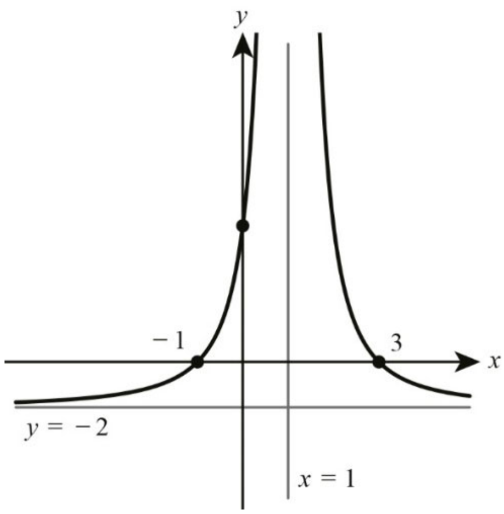
$$\begin{aligned} 4 - 3x &= -8 \\ -3x &= -12 \\ x &= 4 \end{aligned}$$

10 Reflect the graph in the line $y = x$:

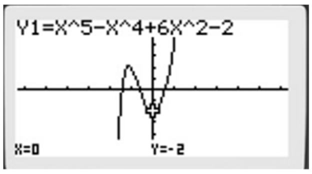


11 Put in the vertical and horizontal asymptotes and the x-intercepts (zeros of the function).

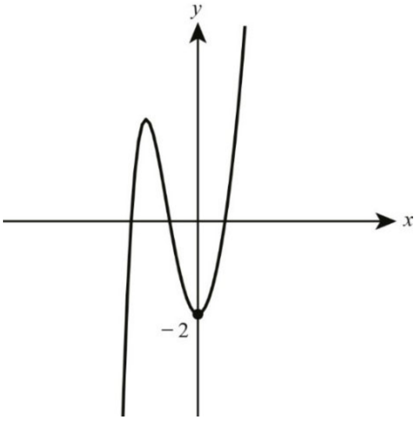
Since $f(x) > -2$, it must tend to ∞ as it approaches the vertical asymptote from either side.



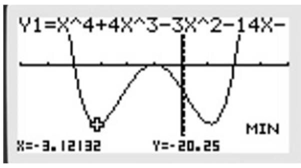
12 Graph the function using the GDC:



The y-intercept is $(0, -2)$. Now sketch the graph from the plot on the GDC:



13 a Graph the function and use 'min' and 'max' to find the coordinates of the vertices, moving the cursor as necessary:

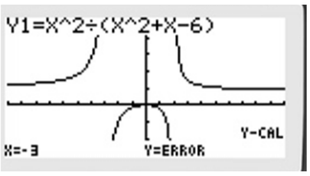


Coordinates of vertices: $(-3.12, -20.3), (-1, 0), (1.12, -20.3)$

b From the graph you can see there is a line of symmetry through the maximum point:

Line of symmetry: $x = -1$

14 Graph the function and look for values where there appear to be asymptotes:



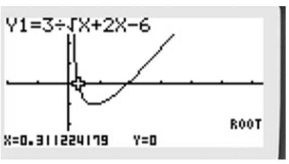
Vertical asymptotes occur at values of x where the y-values appears as 'error':

Vertical asymptotes: $x = -3$ and $x = 2$

The y-value approaches 1 as the x value gets big and positive or big and negative:

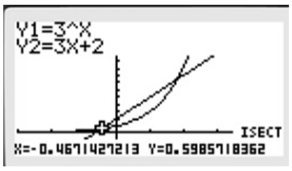
Horizontal asymptote: $y = 1$

15 Graph the function and use 'root', moving the cursor from one to the other:



Zeros: $x = 0.311, 1.92$

16 Graph the function and use ‘isct’, moving the cursor from one intersection point to the other:



Points of intersection: $(-0.467, 0.599)$ and $(1.83, 7.50)$

17 a Substitute $\mathrm{g}(x)$ into $\mathrm{f}(x)$:

$$\begin{aligned}\mathrm{f}\big(\mathrm{g}(x)\big) &= \frac{1}{(3x-4)-2}\\ &= \frac{1}{3x-6}\end{aligned}$$

b Substitute $\mathrm{f}(x)$ into $\mathrm{g}(x)$:

$$\mathrm{g}\big(\mathrm{f}(x)\big) = 3\left(\tfrac{1}{x-2}\right) - 4$$

$$= \frac{3}{x-2} - 4$$

18 Domain of f is $x \le 2$ so domain of fg is

$$\begin{aligned}x - 3 &\le 2\\ x &\le 5\end{aligned}$$

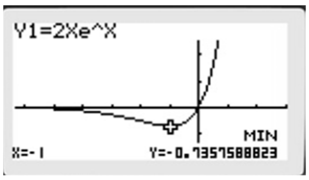
19 Let $y = \mathrm{f}(x)$ and rearrange to make x the subject:

$$\begin{aligned}y &= \frac{x-1}{x+2}\\ xy + 2y &= x - 1\\ x - xy &= 1 + 2y\\ x(1-y) &= 1 + 2y\\ x &= \frac{1+2y}{1-y}\end{aligned}$$

So,

$$\mathrm{f}^{-1}(x) = \frac{1+2x}{1-x}$$

20 Graph the function and use ‘min’ to find the coordinates of the minimum point:



The turning point has x-coordinate $x = -1$, so largest possible domain of given form is $x \ge -1$.

21 Graph **A** is the only negative quadratic so that is equation **b**.

Graph **B** has a negative y-intercept so that is equation **c**.

Graph **C** has a positive y-intercept so that is equation **a**.

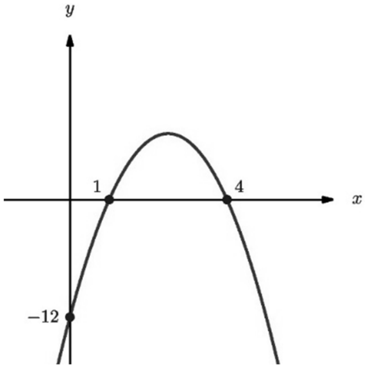
22 Negative quadratic with y-intercept $(0, -12)$

For x-intercepts solve $-3x^2 + 15x - 12 = 0$:

$-3(x^2 - 5x + 4) = 0$

$-3(x - 1)(x - 4) = 0$

$x = 1$ or 4

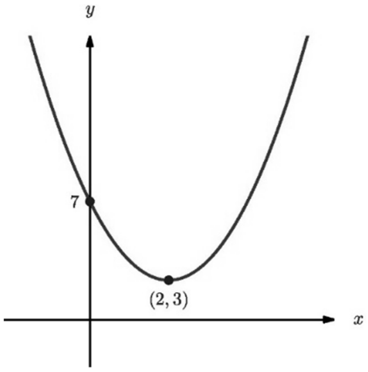


23 a $x^2 - 4x + 7 = (x - 2)^2 - 4 + 7$

$= (x - 2)^2 + 3$

b Positive quadratic with y-intercept $(0, 7)$

Vertex at $(h, k) = (2, 3)$



24 $2x^2 + 7x - 15 = 0$

$(2x - 3)(x + 5) = 0$

So

$2x - 3 = 0$ so $x = \frac{3}{2}$

or

$x + 5 = 0$ so $x = -5$

25 a $x^2 - 5x + 3 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 3$

$$= \left(x - \frac{5}{2}\right)^2 - \frac{13}{4}$$

b Use the completed square form from part **a** and solve for x:

$$x^2 - 5x + 3 = 0$$

$$\left(x - \frac{5}{2}\right)^2 - \frac{13}{4} = 0$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{13}{4}$$

$$x - \frac{5}{2} = \pm\frac{\sqrt{13}}{2}$$

$$x = \frac{5 \pm \sqrt{13}}{2}$$

26 Use the quadratic formula with $a = 3, b = -4, c = -2$:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{40}}{6}$$

$$= \frac{4 \pm 2\sqrt{10}}{6}$$

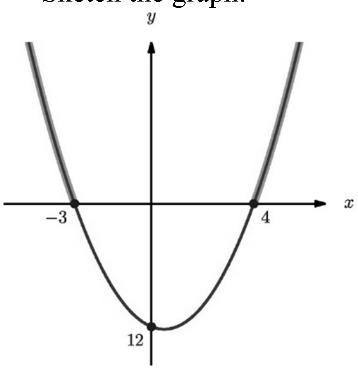
$$= \frac{2 \pm \sqrt{10}}{3}$$

27 Solve the equation $x^2 - x - 12 = 0$:

$$(x - 4)(x + 3) = 0$$

$$x = 4 \text{ or } -3$$

Sketch the graph:



So, $x < -3$ or $x > 4$

28 $\Delta = 5^2 - 4(4)(3)$

$= 25 - 48$

$= -23 < 0$

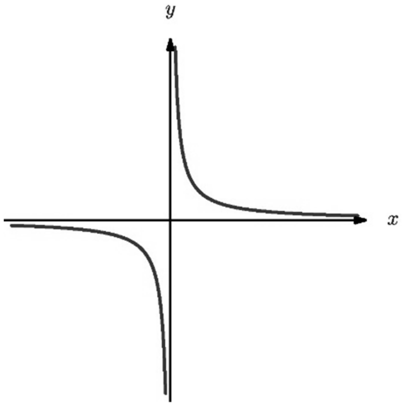
So, no real roots

29 Two distinct real roots, so $\Delta > 0$ (where $a = 3k, b = 4, c = 12k$)

$$4^2 - 4(3k)(12k) > 0$$
$$4 - 36k^2 > 0$$
$$k^2 < \frac{1}{9}$$
$$-\frac{1}{3} < k < \frac{1}{3}$$

30 Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

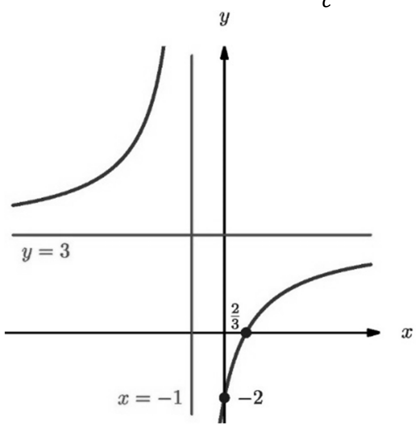


31 $y = \frac{ax+b}{cx+d}$ has

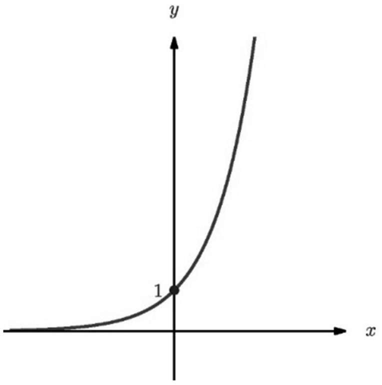
x-intercept $\left(-\frac{b}{a}, 0\right)$ y-intercept $\left(0, \frac{b}{d}\right)$

Vertical asymptote $x = -\frac{d}{c}$

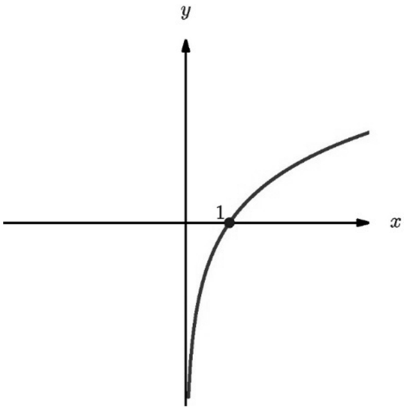
Horizontal asymptote $y = \frac{a}{c}$



32

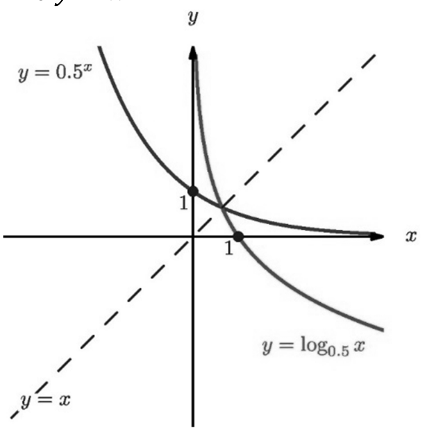


33



34 $y = 0.5^x$ is decreasing since $0.5 < 1$.

$y = 0.5^x$ and $y = \log_{0.5} x$ are inverse functions, so one is a reflection of the other in the line $y = x$



35 $2.8^x = e^{x \ln 2.8}$
$= e^{1.03x}$

So, $k = 1.03$

36 Rearrange so RHS is 0 and then factorise:

$$x\ln x - 4x = 0$$
$$x(\ln x - 4) = 0$$

So

$$x = 0$$

or

$$\ln x = 4$$
$$x = e^4$$

37 Let $y = \sqrt{x}$

$$y^2 - 7y + 10 = 0$$
$$(y-2)(y-5) = 0$$
$$y = 2 \text{ or } 5$$

So,

$$\sqrt{x} = 2 \text{ or } 5$$
$$x = 4 \text{ or } 25$$

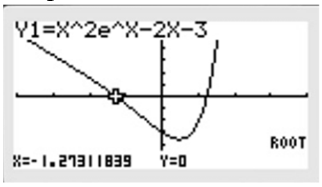
38 Combine the log terms using $\log x + \log y = \log xy$ and then undo the log leaving a quadratic equation:

$$\log_2\big(x(x+2)\big) = 3$$
$$x^2 + 2x = 2^3$$
$$x^2 + 2x - 8 = 0$$
$$(x+4)(x-2) = 0$$
$$x = -4 \text{ or } 2$$

However, checking both possible solutions in the original equation, you can see that $x = -4$ is not valid, as you cannot have a log of a negative number.

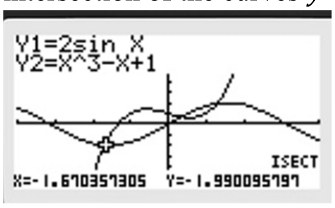
So, $x = 2$.

39 Graph the function and use ‘root’, moving the cursor from one to the other:



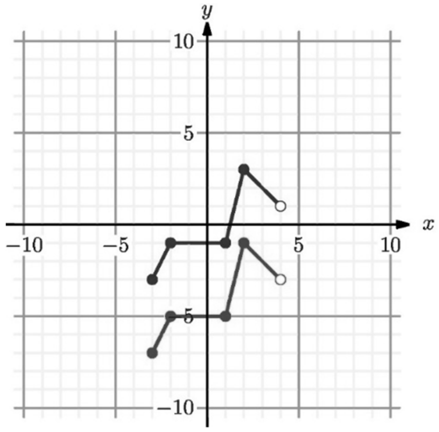
$x = -1.27$ or 1.25

40 This could be solved as above by rearranging to the form $f(x) = 0$ or by finding the intersection of the curves $y = 2\sin x$ and $y = x^3 - x + 1$:



$x = -1.67, 0.353$ or 1.31

41 $y = f(x) - 4$ is a vertical translation by -4:



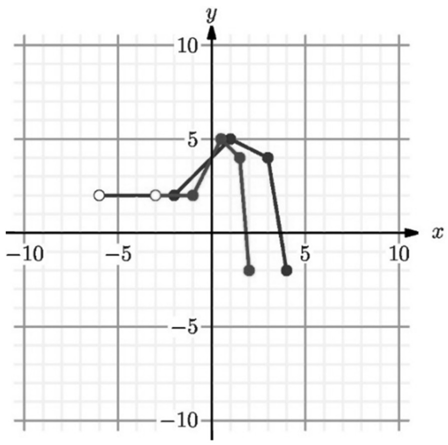
42 A translation 3 units to the right is given by $y = f(x - 3)$:

$$\begin{aligned} y &= (x-3)^2 - 2(x-3) + 5 \\ &= x^2 - 6x + 9 - 2x + 6 + 5 \\ &= x^2 - 8x + 20 \end{aligned}$$

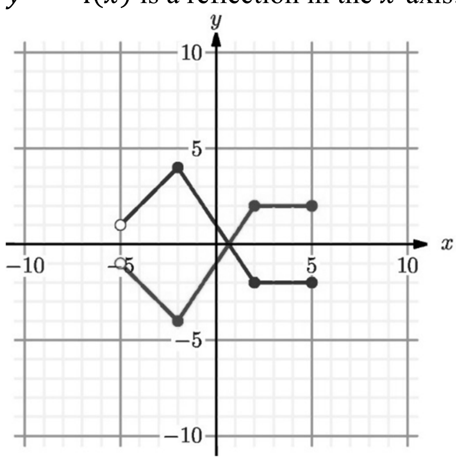
43 A vertical stretch by scale factor 2 is given by $y = 2f(x)$:

$$\begin{aligned} y &= 2(3x^2 + x - 2) \\ &= 6x^2 + 2x - 4 \end{aligned}$$

44 $y = f(2x)$ is a horizontal stretch with scale factor $\frac{1}{2}$:



45 $y = -f(x)$ is a reflection in the x-axis:



46 A reflection in the y-axis is given by $y = \mathrm{f}(-x)$:

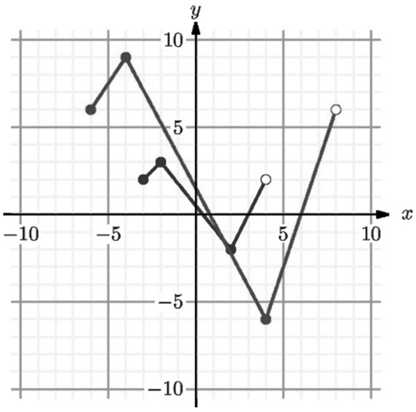
$$y = (-x)^3 + 3(-x)^2 - 4(-x) + 1$$
$$= -x^3 + 3x^2 + 4x + 1$$

47 $y = 4\mathrm{f}(x) + 1$ is a vertical stretch with scale factor 4 followed by a vertical translation by 1.

So, $(3, -2) \longrightarrow (3, -2 \times 4 + 1) = (3, -7)$

Note that the order matters for two vertical transformations.

48 $y = 3\mathrm{f}\left(\frac{1}{2}x\right)$ is vertical stretch with scale factor 3 and a horizontal stretch with scale factor 2 (in either order):



49 Graph **A** is a polynomial with even degree since for large positive and large negative values of x the behaviour is the same.

Graphs **B** and **C** are both polynomials with odd degree since in each case the behaviour for large positive values of x is opposite to that for large negative values of x.

Since graph **B** is large and positive when x is large and negative, the highest power must have a negative coefficient.

So,graph **A** has equation **b**

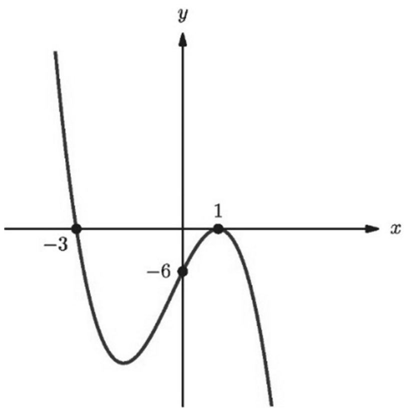
graph **B** has equation **c**

graph **C** has equation **a**

50 The term with the highest power of x is $-2x^3$ so the graph has negative cubic shape.

There are roots at $x = 1$ and $x = -3$ but since the factor $(x - 1)$ is squared, the graph only touches the x-axis at $x = 1$.

The y-intercept is $y = -2(0 - 1)^2(0 + 3) = -6$.



51 The graph has factors $(x-4)$ and $(x+1)$ corresponding to the roots $x=4$ and $x=-1$ respectively. Since the gradient at $x=-1$ is zero, the factor $(x+1)$ is cubed:

$y=a(x-4)(x+1)^3$

When $x=0, y=12$:

$a(0-4)(0+1)^3=12$
$-4a=12$
$a=-3$

So, $y=-3(x-4)(x+1)^3$

52 The remainder is given by $\mathrm{f}\left(\frac{-3}{2}\right)$:

$r=2\left(\frac{-3}{2}\right)^3-\left(\frac{-3}{2}\right)^2+4\left(\frac{-3}{2}\right)+3$
$=-12$

53 If $(3x-4)$ is a factor then $\mathrm{f}\left(\frac{4}{3}\right)=0$:

$3\left(\frac{4}{3}\right)^3+5\left(\frac{4}{3}\right)^2-42\left(\frac{4}{3}\right)+a=0$
$-40+a=0$
$a=40$

54 a Use the factor theorem:

$\mathrm{f}(-2)=3(-2)^3+22(-2)^2+20(-2)-24$
$=0$

Therefore, $(x+2)$ is a factor of $\mathrm{f}(x)$.

b $\mathrm{f}(x)$ is the product of $(x+2)$ and a quadratic factor of the form $3x^2+ax-12$: $3x^3+22x^2+20x-24=(x+2)(3x^2+ax-12)$

Equating coefficients of x^2:

$22=a+6$
$a=16$

So, $\mathrm{f}(x)=(x+2)(3x^2+16x-12)$

Factorize the quadratic and solve:

$(x+2)(3x^2+16x-12)=0$
$(x+2)(3x-2)(x+6)=0$
$x=-2, \frac{2}{3}, -6$

55 Use the results sum $=-\frac{a_{n-1}}{a_n}$ and product $=\frac{(-1)^n a_0}{a_n}$:

Sum $=-\frac{55}{6}$

Product $=\frac{(-1)^4(-80)}{6}=-\frac{40}{3}$

56 First find the sum and product of the roots α and β:

$\alpha + \beta = -\frac{-2}{4} = \frac{1}{2}$

$\alpha\beta = \dfrac{3}{4}$

Then use these to find the sum and product of the roots $\frac{2}{\alpha}$ and $\frac{2}{\beta}$:

$\frac{2}{\alpha} + \frac{2}{\beta} = \frac{2(\alpha+\beta)}{\alpha\beta} = \frac{2\left(\frac{1}{2}\right)}{\frac{3}{4}} = \frac{4}{3}$

$\frac{2}{\alpha} \times \frac{2}{\beta} = \frac{4}{\alpha\beta} = \frac{4}{\frac{3}{4}} = \frac{16}{3}$

Finally relate these to the coefficients of the new quadratic $ax^2 + bx + c = 0$:

$-\frac{b}{a} = \frac{4}{3} \Rightarrow b = -\frac{4a}{3}$

$\frac{c}{a} = \frac{16}{3} \Rightarrow c = \frac{16a}{3}$

Let $a = 3$, so that $b = -4$ and $c = 16$

So, the equation is $3x^2 - 4x + 16 = 0$

57 The sum of the roots will be $-\frac{a_3}{a_4}$:

$\frac{1}{2} - 3 + 2 + \mathrm{i} + 2 - \mathrm{i} = -\frac{-3}{a}$

$\dfrac{3}{2} = \dfrac{3}{a}$

$a = 2$

The product of the roots will be $\frac{(-1)^4 a_0}{a_4}$:

$\left(\frac{1}{2}\right)(-3)(2+\mathrm{i})(2-\mathrm{i}) = \frac{c}{2}$

$-\dfrac{3}{2}(5) = \dfrac{c}{2}$

$c = 15$

58 $y = \frac{ax+b}{cx^2+dx+e}$ has

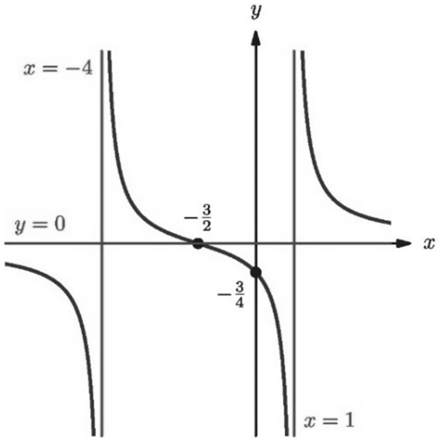
- x-intercept $\left(-\frac{b}{a}, 0\right)$
- y-intercept $\left(0, \frac{b}{e}\right)$
- Horizontal asymptote $y = 0$
- Vertical asymptotes for any real roots of $cx^2 + dx + e = 0$

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0$

$x = -4, 1$

So, vertical asymptotes at $x = -4$ and $x = 1$



59 $y = \frac{ax^2+bx+c}{dx+e}$ has

- x-intercepts for any real roots of $ax^2 + bx + c = 0$
- y-intercept $\left(0, \frac{c}{e}\right)$
- Vertical asymptote $x = -\frac{e}{d}$
- An oblique asymptote of the form $y = px + q$

$x^2 + x - 12 = 0$

$(x - 3)(x + 4) = 0$

$x = 3, -4$

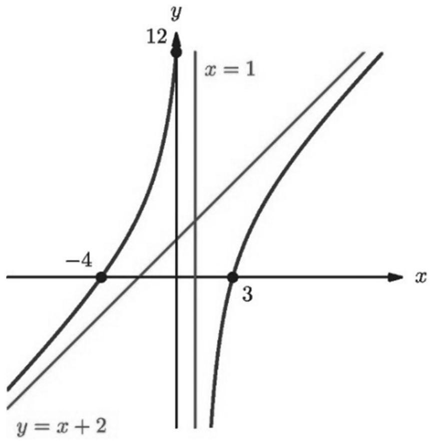
So, x-intercepts $(3, 0)$ and $(-4, 0)$

Divide the rational function out to find the oblique asymptote:

$x^2 + x - 12 = (x - 1)(x + 2) - 10$

$\frac{x^2+x-12}{x-1} = x + 2 - \frac{10}{x-1}$

So, the oblique asymptote is $y = x + 2$



60 Find $\mathrm{f}(-x)$ and see whether it is the same as $-\mathrm{f}(x), \mathrm{f}(x)$ or neither:

$$\mathrm{f}(-x) = \frac{\cos(-x)}{-x}$$
$$= -\frac{\cos x}{x}$$
$$= -\mathrm{f}(x)$$

So, $\mathrm{f}(x)$ is an odd function.

61 Graph **A** is not symmetric in the origin and not symmetric in the y-axis, so it is neither odd nor even

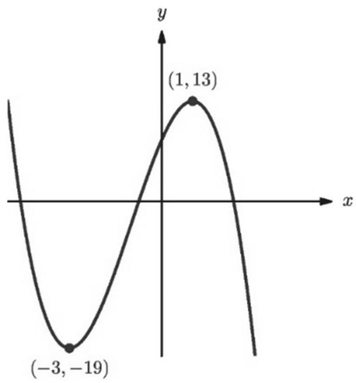
Graph **B** is symmetric in the y-axis so it is an even function

Graph **C** is symmetric in the origin so it is an odd function

62 Find the x-values of the turning points:

$$-3x^2 - 6x + 9 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$x = -3, 1$$

Sketch the graph:



The function is one-to-one between the two turning points so this is the largest interval of the required form:

So, $-3 \le x \le 1$

63 Find $\mathrm{f}^{-1}(x)$:

$$y = \frac{3x-1}{x-3}$$
$$xy - 3y = 3x - 1$$
$$xy - 3x = 3y - 1$$
$$x = \frac{3y-1}{y-3}$$
$$\mathrm{f}^{-1}(x) = \frac{3x-1}{x-3}$$

$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ and so f is self-inverse.

64 Graph **A** isn't symmetric in the line $y = x$ so it isn't self-inverse

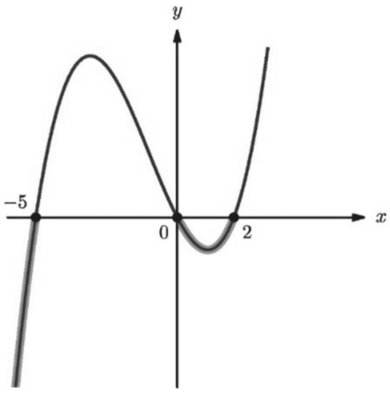
Graph **B** isn't symmetric in the line $y = x$ so it isn't self-inverse

Graph **C** is symmetric in the line $y = x$ so it is self-inverse

65 Rearrange so that $x^3 + 3x^2 - 10x \leq 0$ and solve the equation $x^3 + 3x^2 - 10x = 0$:

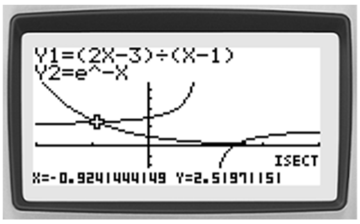
$x(x + 5)(x - 2) = 0$
$x = 0, -5, 2$

Sketch the graph:



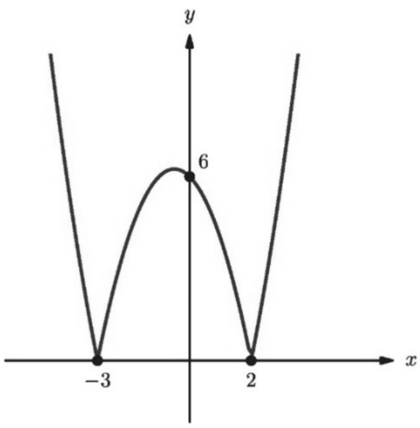
So, $x \leq -5$ or $0 \leq x \leq 2$

66 Graph $y = \frac{2x-3}{x-1}$ and $y = \mathrm{e}^{-x}$ using the GDC and find the points of intersection:

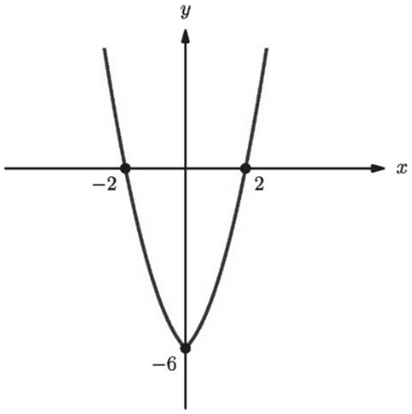


Read off the required region (noting the asymptote at $x = 1$):
So, $-0.924 < x < 1$ or $x > 1.56$

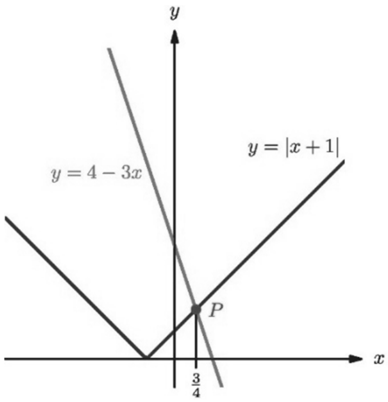
67 Sketch the graph of $y = x^2 + x - 6$ and reflect the part below the x-axis to be above the x-axis:



68 Sketch the graph of $y = x^2 + x - 6$ for $x \geq 0$ and reflect that in the y-axis:



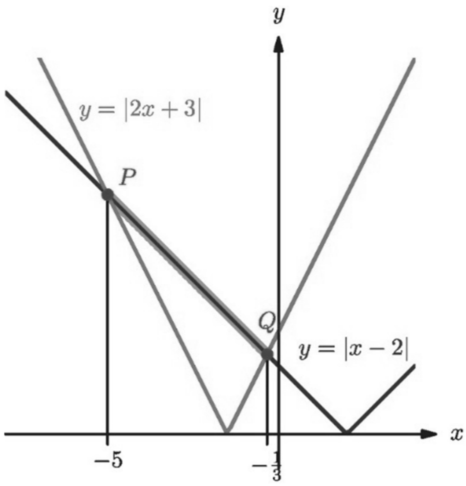
69 Sketch the graphs of $y = |x + 1|$ and $y = 4 - 3x$ on the same axes:



There is just one intersection point P (solution to the equation), which is on the original part of $y = |x + 1|$:

$x + 1 = 4 - 3x$
$4x = 3$
$x = \dfrac{3}{4}$

70 Sketch the graphs of $y = |x - 2|$ and $y = |2x + 3|$ on the same axes:



The graphs intersect at two points, P and Q, both of which are on the reflected part of $y = |x - 2|$:

At Q $\quad -(x - 2) = 2x + 3$

$-x + 2 = 2x + 3$

$3x = -1$

$x = -\dfrac{1}{3}$

At P $\quad -(x - 2) = -(2x + 3)$

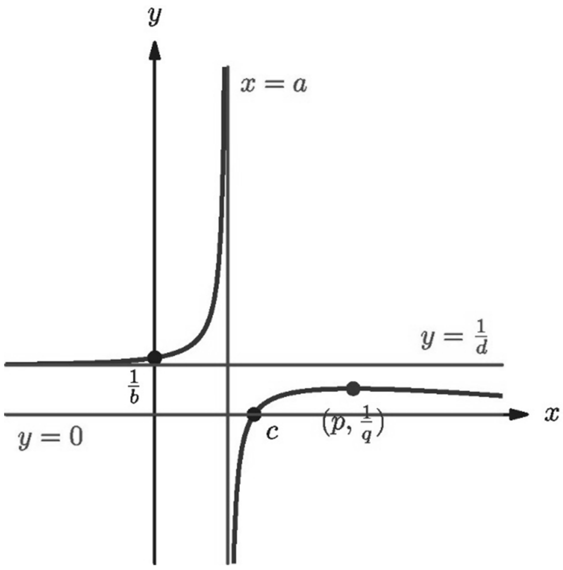
$x - 2 = 2x + 3$

$x = -5$

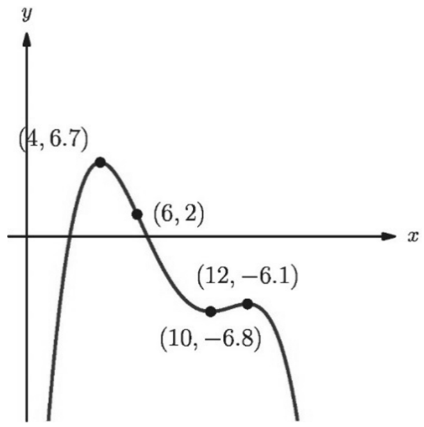
So, $-5 < x < -\frac{1}{3}$

71 $y = f(x)$ has

- an x-intercept at $(a, 0)$ so $y = \frac{1}{f(x)}$ has a vertical asymptote at $x = a$
- a y-intercept at $(0, b)$ so $y = \frac{1}{f(x)}$ has a y-intercept at $\left(0, \frac{1}{b}\right)$
- a vertical asymptote at $x = c$ so $y = \frac{1}{f(x)}$ has an x-intercept at $(c, 0)$
- a horizontal asymptote at $y = d$ so $y = \frac{1}{f(x)}$ has a horizontal asymptote at $y = \frac{1}{d}$
- a minimum point at (p, q) so $y = \frac{1}{f(x)}$ has a maximum point at $\left(p, \frac{1}{q}\right)$
- $y \to \infty$ as $x \to \infty$ so $y = \frac{1}{f(x)}$ has a horizontal asymptote at $y = 0$



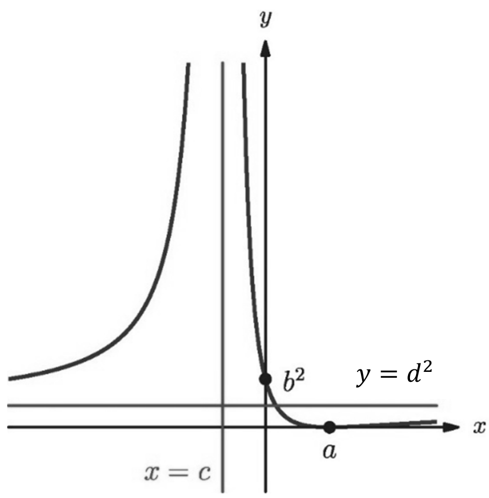
72 $y = \mathrm{f}\left(\frac{1}{2}x - 3\right)$ is a horizontal translation by 3 followed by a horizontal stretch with scale factor $\frac{1}{\frac{1}{2}} = 2$:



73 $y = f(x)$ has

- an x-intercept at $(a, 0)$ so $y = [f(x)]^2$ has a minimum point at $(a, 0)$
- a y-intercept at $(0, -b)$ so $y = [f(x)]^2$ has a y-intercept at $(0, b^2)$
- a vertical asymptote at $x = c$ so $y = [f(x)]^2$ also has a vertical asymptote at $x = c$
- a horizontal asymptote at $y = d$ so $y = [f(x)]^2$ has a horizontal asymptote at $y = d^2$

All negative values on $y = f(x)$ become positive on $y = [f(x)]^2$:



3 Geometry and trigonometry

1 $d = \sqrt{(7-2)^2 + (3-(-4))^2 + (-1-5)^2}$
$= \sqrt{25 + 49 + 36}$
$= 10.5$

2 $M = \left(\frac{1+(-5)}{2}, \frac{8+2}{2}, \frac{-3+4}{2}\right)$
$= (-2, 5, 0.5)$

3 Radius = 8 cm

Volume $= \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \times 8^3$

$= 2140 \text{ cm}^3$

Surface area $= 4\pi r^2$

$= 4\pi \times 8^2$

$= 804 \text{ cm}^2$

4 Volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times 6^2 \times 15$

$= 565 \text{ cm}^3$

Slope length, l, is given by:

$l = \sqrt{r^2 + h^2}$
$= \sqrt{6^2 + 15^2}$
$= 3\sqrt{29}$

Surface area $= \pi r l + \pi r^2$

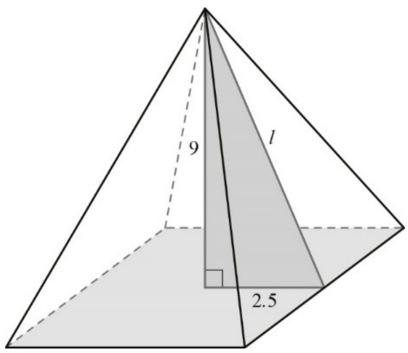
$= \pi \times 6 \times 3\sqrt{29} + \pi \times 6^2$

$= 418 \text{ cm}^2$

5 Volume $= \frac{1}{3}x^2 h$

$= \frac{1}{3} \times 5^2 \times 9$

$= 75.0 \text{ cm}^3$



$l = \sqrt{2.5^2 + 9^2}$
$= \frac{\sqrt{349}}{2}$

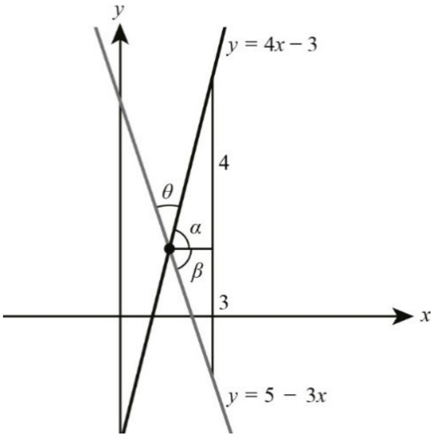
Surface area $= 5^2 + 4\left(\frac{1}{2} \times 5 \times \frac{\sqrt{349}}{2}\right)$

$= 118 \text{ cm}^2$

6 Volume $= \pi r^2 h + \frac{2}{3}\pi r^3$

$$= \pi \times 5^2 \times 30 + \frac{2}{3}\pi \times 5^3$$
$$= 2620 \text{ m}^3$$

7 Draw the lines and label the angle required θ:



Since the gradient of $\boldsymbol{y = 4x - 3}$ is $\mathbf{4}$,

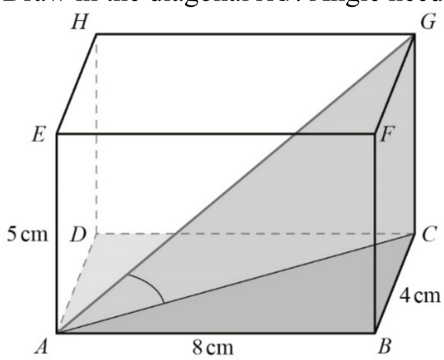
$$\tan\alpha = \frac{4}{1}$$
$$\alpha = \tan^{-1} 4 = 76.0°$$

Since the gradient of $y = 5 - 3x$ is -3,

$$\tan\beta = \frac{3}{1}$$
$$\alpha = \tan^{-1} 3 = 71.6°$$

So, $\theta = 180 - 76.0 - 71.6 = 32.4°$

8 Draw in the diagonal AG. Angle needed is $G\hat{A}C = \theta$



First work in triangle ABC:

$$AC = \sqrt{8^2 + 4^2}$$
$$= \sqrt{80}$$

Then in triangle ACG:

$$\tan\theta = \frac{5}{\sqrt{80}}$$
$$\theta = \tan^{-1}\frac{5}{\sqrt{80}} = 29.2°$$

9 Draw in the two diagonals – they will intersect at the midpoint of each, M.

From triangle AGC:

$$AG = \sqrt{\left(\sqrt{80}\right)^2 + 5^2}$$
$$= \sqrt{105}$$

By symmetry, $EC = \sqrt{105}$

And $AM = CM = \frac{\sqrt{105}}{2}$

Using the cosine rule in triangle AMC:

$$\cos M = \frac{AM^2 + CM^2 - AC^2}{2(AM)(CM)}$$
$$= \frac{26.25 + 26.25 - 80}{52.5}$$
$$M = 121.6°$$

So, acute angle between AG and EC is $180 - 121.6 = 58.4°$

10 $\sin\theta = \frac{1.8}{4.9}$

$$\theta = \sin^{-1}\frac{1.8}{4.9}$$
$$= 21.6°$$

11 By the sine rule,

$$\frac{3.8}{\sin 80} = \frac{AC}{\sin 55}$$
$$AC = \frac{3.8}{\sin 80} \times \sin 55$$
$$= 3.16 \text{ cm}$$

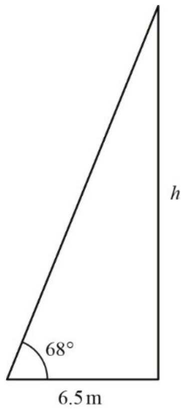
12 By the cosine rule,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$= \frac{10^2 + 12^2 - 9^2}{2 \times 10 \times 12}$$
$$C = \cos^{-1}\left(\frac{10^2 + 12^2 - 9^2}{2 \times 10 \times 12}\right) = 47.2°$$

13 $A = \frac{1}{2} \times 6 \times 15 \times \sin 42$

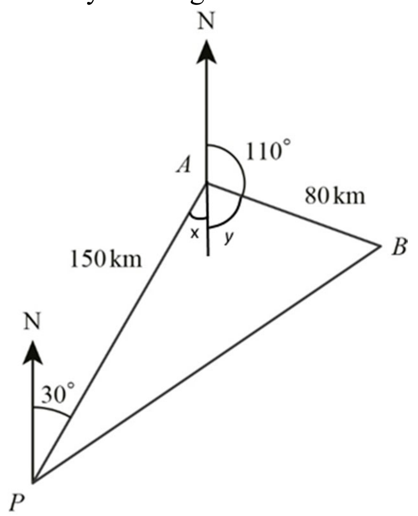
$= 30.1$

14 Draw a diagram:



$$\tan 68 = \frac{h}{6.5}$$
$$h = 6.5 \tan 68$$
$$= 16.1 \text{ m}$$

15 Start by drawing the situation described:



$x = 30°$ by alternate angles

$y = 180 - 110 = 70°$

So, $P\hat{A}B = 30 + 70 = 100°$

By the cosine rule,

$$d^2 = 150^2 + 80^2 - 2 \times 150 \times 80 \times \cos 100$$
$$d = \sqrt{150^2 + 80^2 - 2 \times 150 \times 80 \times \cos 100}$$
$$= 182 \text{ km}$$

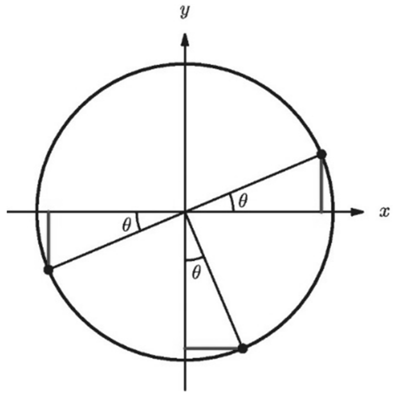
16 a $55° = 55 \times \frac{2\pi}{360} = \frac{11\pi}{36}$ radians

b 1.2 radians $= 1.2 \times \frac{360}{2\pi} = 68.8°$

17 $s = r\theta$
$= 6 \times 0.7$
$= 4.2$ cm

18 $A = \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 10^2 \times 1.8$
$= 90 \text{ cm}^2$

19



Using the unit circle:

a $\sin(\theta + \pi) = -\sin\theta = -0.4$

b $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta = 0.4$

20 $\tan(2\pi - \theta) = \frac{\sin(2\pi-\theta)}{\cos(2\pi-\theta)}$

$$= \frac{\sin(-\theta)}{\cos(-\theta)} \quad \text{(since sin and cos are } 2\pi \text{ periodic)}$$

$$= \frac{-\sin\theta}{\cos\theta}$$

$$= -\tan\theta$$

Note that certain relationships, such as $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, are used so often that you should just know them. You don't want to have to go back to the unit circle each time to derive them.

21 Relate $\frac{4\pi}{3}$ to one of the 'standard' angles: $\cos\frac{\pi}{3} = \frac{1}{2}$

$$\cos\frac{4\pi}{3} = \cos\left(\frac{\pi}{3} + \pi\right)$$

$$= -\cos\frac{\pi}{3} \quad \text{(using the unit circle or remembering } \cos(\theta + \pi) = -\cos\theta\text{)}$$

$$= -\frac{1}{2}$$

22 By the sine rule,

$$\frac{\sin\theta}{14} = \frac{\sin 38}{11}$$

$$\theta = \sin^{-1}\left(\frac{\sin 38}{11} \times 14\right)$$

$\theta = 51.6°$ or $\theta = 180 - 51.6 = 128.4°$

Check that each value of θ is possible by making sure that the angle sum in each case is less than 180°:

$51.6 + 38 = 89.6 < 180$

$128.4 + 38 = 166.4 < 180$

So both are possible:

$\theta = 51.6°$ or $128.4°$

23 Use the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to relate the value of $\cos\theta$ to the value of $\sin\theta$:

$$\cos^2\theta = 1 - \sin^2\theta$$
$$= 1 - \frac{9}{16}$$
$$= \frac{7}{16}$$

$$\cos\theta = \pm\frac{\sqrt{7}}{4}$$

But $\cos\theta < 0$ for $\frac{\pi}{2} < \theta < \pi$

So, $\quad \cos\theta = -\frac{\sqrt{7}}{4}$

24 $(\cos\theta + \sin\theta)^2 \equiv \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$
$$\equiv 2\sin\theta\cos\theta + \cos^2\theta + \sin^2\theta$$
$$= \sin 2\theta + 1$$

25 Relate 15° to a ‘standard’ angle:

$$\cos(2\times 15) = \cos 30 = \frac{\sqrt{3}}{2}$$

So, use $\cos 2\theta = 2\cos^2\theta - 1$:

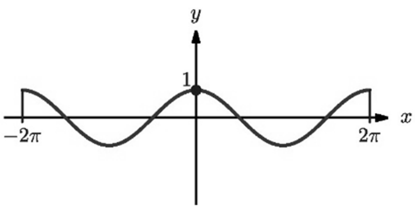
$$\cos 30 = 2\cos^2 15 - 1$$

$$\cos^2 15 = \frac{\cos 30 + 1}{2}$$
$$= \frac{\frac{\sqrt{3}}{2} + 1}{2}$$
$$= \frac{\sqrt{3} + 2}{4}$$

So, taking the positive square root (since $\cos 15° > 0$):

$$\cos 15° = \sqrt{\frac{\sqrt{3}+2}{4}}$$

26

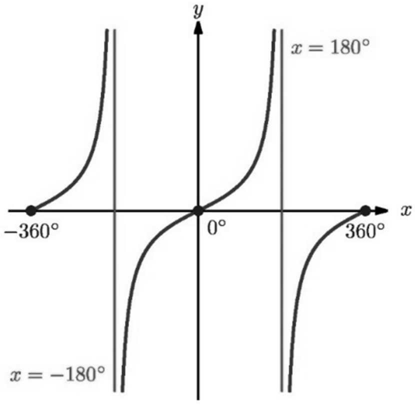


Period $= 2\pi$

Amplitude $= 1$

27 $y = \tan\frac{x}{2}$ is a transformation of the form $y = \mathrm{f}\left(\frac{1}{2}x\right)$ where $\mathrm{f}(x) = \tan x$

So, it is a horizontal stretch of the tan curve with scale factor 2.



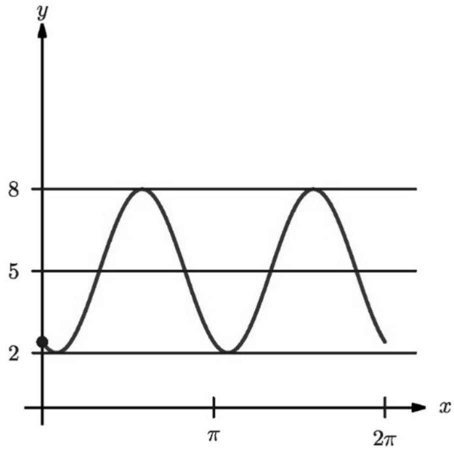
28 $y = 3\sin\theta + 5$ is a transformation of the form $y = 3\mathrm{f}(\theta) + 5$ where $\mathrm{f}(\theta) = \sin\theta$

So, vertically there is a stretch with scale factor 3 followed by a translation by 5 of the sin curve.

Horizontally, max points of $\sin\theta$ occur at $\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \ldots$

So,

$$2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}.$$
$$x - \frac{\pi}{3} = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4} \ldots$$
$$x = \frac{7\pi}{12}, \frac{19\pi}{12}$$



29 The amplitude is half the difference between the minimum and maximum heights:

$$a = \frac{0.5 - 0.16}{2} = 0.17$$

The period is twice the time from the maximum height to the minimum height:

Period $= 2 \times 0.3 = 0.6$

But period $= \frac{2\pi}{b}$

So,

$$0.6 = \frac{2\pi}{b}$$
$$b = \frac{10\pi}{3}$$

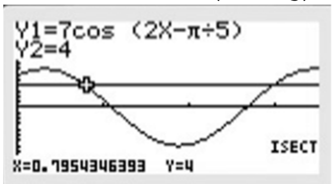
c is halfway between the maximum and minimum heights:

$$c = \frac{0.5 + 0.16}{2} = 0.33$$

So,

$$h = 0.17\cos\left(\frac{10\pi}{3}t\right) + 0.33$$

30 Graph $y = 7\cos\left(2x - \frac{\pi}{5}\right)$ and $y = 4$ and find the x-values of the intersection points:

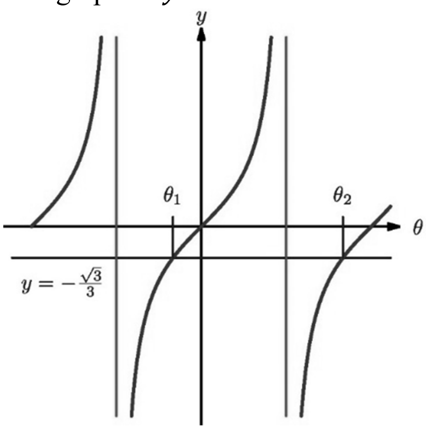


$x = 0.795, 2.97$

31 Since $\tan\frac{\sqrt{3}}{3} = \frac{\pi}{6}$ and $\tan(-\theta) = -\tan\theta$,

$$\theta_1 = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

The graph of $y = \tan\theta$ for $-\pi < \theta < \pi$ shows there is one other solution:



$\theta_2 = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$ So, $\theta = -\frac{\pi}{6}, \frac{5\pi}{6}$

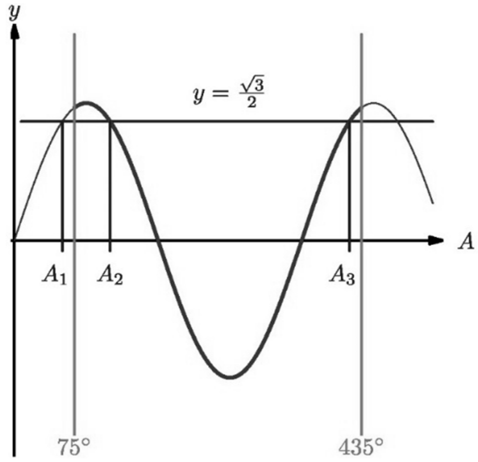
32 Let $A = x + 75$

Then, $0 < x < 360 \Rightarrow 75 < x + 75 < 360 + 75$.

So, $\quad \sin A = \frac{\sqrt{3}}{2} \quad$ for $75° < A < 435°$

$A_1 = \sin^{-1}\frac{\sqrt{3}}{2} = 60°$

The graph of $y = \sin A$ for $75° < A < 435°$ shows there are two solutions (A_2 and A_3):



$A_2 = 180 - 60 = 120°$
$A_3 = 60 + 360 = 420°$

So, $A = 120°, 420°$

$x = 45°, 345°$

33 Use the sine double angle formula so that everything is a function of θ:

$$\sin 2\theta = \sin\theta$$
$$2\sin\theta\cos\theta = \sin\theta$$
$$2\sin\theta\cos\theta - \sin\theta = 0$$
$$\sin\theta\,(2\cos\theta - 1) = 0$$

So,

$\sin\theta = 0$

$\theta = 0, \pi, 2\pi$

or

$2\cos\theta - 1 = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

Note that while it is tempting to cancel $\sin\theta$ on the second line of the working, this would lose the solutions resulting from $\sin\theta = 0$.

34 Use $\sin^2 x \equiv 1 - \cos^2 x$ so that only $\cos x$ is involved:

$$\begin{aligned} 2\sin^2 x - 3\cos x - 3 &= 0 \\ 2(1 - \cos^2 x) - 3\cos x - 3 &= 0 \\ 2 - 2\cos^2 x - 3\cos x - 3 &= 0 \\ 2\cos^2 x + 3\cos x + 1 &= 0 \end{aligned}$$

This is a quadratic in $\cos x$:

$(2\cos x + 1)(\cos x + 1) = 0$

So

$$\begin{aligned} \cos x &= -\frac{1}{2} \\ x &= 120°, -120° \end{aligned}$$

or

$$\begin{aligned} \cos x &= -1 \\ x &= 180°, -180° \end{aligned}$$

$x = -180°, -120°, 120°, 180°$

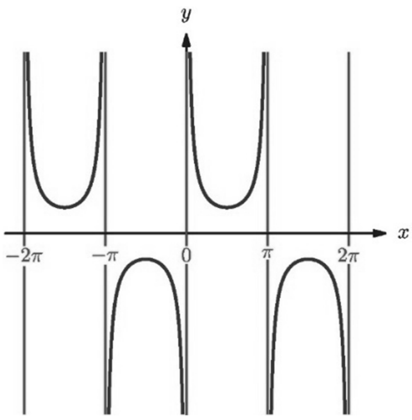
35 Use the definition $\sec\theta = \frac{1}{\cos\theta}$:

$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}}$

$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)}$

$= \sqrt{2}$

36



37 Use the identity $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$ to relate the value of $\tan\theta$ to the value of $\sin\theta$:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\left(-\frac{2}{3}\right)} = -\frac{3}{2}$$

$$\operatorname{cosec}^2\theta = 1 + \left(-\frac{3}{2}\right)^2$$
$$= 1 + \frac{9}{4}$$
$$= \frac{13}{4}$$

$$\operatorname{cosec}\theta = \pm\frac{\sqrt{13}}{2}$$

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} = \pm\frac{2}{\sqrt{13}}$$

But, $\sin\theta < 0$ for $\frac{3\pi}{2} < \theta < 2\pi$

So, $\quad \sin\theta = -\frac{2}{\sqrt{13}}$

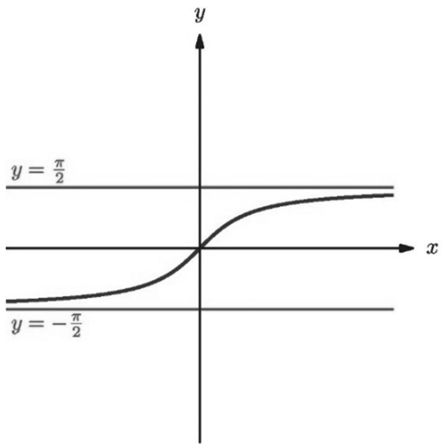
38 $y = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$

$$\sin y = -\frac{\sqrt{3}}{2}$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y = -\frac{\pi}{3}$

So, $\quad \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

39



Domain: $x \in \mathbb{R}$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

40 Use the compound angle identity $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$ with the standard angles $A = 60°$ and $B = 45°$:

$$\sin 105° = \sin(60° + 45°)$$
$$= \sin 60° \cos 45° + \cos 60° \sin 45°$$
$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

41 $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$$-\frac{4}{3} = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$4\tan^2\theta - 4 = 6\tan\theta$$
$$2\tan^2\theta - 3\tan\theta - 2 = 0$$
$$(2\tan\theta + 1)(\tan\theta - 2) = 0$$
$$\tan\theta = -\frac{1}{2} \text{ or } 2$$

42 $\cos(\pi - x) = \cos\pi\cos x + \sin\pi\sin x$

$$= (-1)\cos x + 0 \times \sin x$$
$$= -\cos x$$

43 The vector goes 5 units to the left and 2 units up:

$$\mathbf{v} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

44 $\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

45 $\mathbf{i} - 6\mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix}$

46 Trace a path from P to S via Q and R, noting that going backwards along an arrow means the vector needs to be negative:

$$\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS}$$
$$= \mathbf{a} + \mathbf{b} - \mathbf{c}$$

47 Since M is the midpoint of YZ, $\overrightarrow{YM} = \frac{1}{2}\overrightarrow{YZ}$.

So, start by finding an expression for $\overrightarrow{YZ}$:

$$\overrightarrow{YZ} = \overrightarrow{YX} + \overrightarrow{XZ}$$
$$= -\mathbf{a} + \mathbf{b}$$

$$\overrightarrow{XM} = \overrightarrow{XY} + \frac{1}{2}\overrightarrow{YZ}$$
$$= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b})$$
$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

48 If two vectors are parallel, then $\mathbf{b} = t\mathbf{a}$ for some scalar t:

$$\begin{pmatrix} p \\ q \\ -12 \end{pmatrix} = t\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} -t \\ 2t \\ 4t \end{pmatrix}$$

$$\begin{cases} p = -t & (1) \\ q = 2t & (2) \\ -12 = 4t & (3) \end{cases}$$

From (3): $t = -3$

So, $p = 3, q = -6$

49 $|\mathbf{v}| = \sqrt{3^2 + (-5)^2 + (-1)^2}$
$= \sqrt{35}$

50 Unit vector $= \frac{1}{\sqrt{2^2+6^2+(-3)^2}}\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$

$$= \frac{1}{7}\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$$

51 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
$$= \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} -9 \\ 2 \\ 1 \end{pmatrix}$$

52 $AB = |\mathbf{b} - \mathbf{a}|$
$= |(5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) - (\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})|$
$= |4\mathbf{i} + \mathbf{j} - 4\mathbf{k}|$
$= \sqrt{4^2 + 1^2 + (-4)^2}$
$= \sqrt{33}$

53 For $ABCD$ to be a parallelogram, need $\overrightarrow{AB} = \overrightarrow{DC}$:

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
$= (5\mathbf{i} - 4\mathbf{j} + \mathbf{k}) - (3\mathbf{i} - 2\mathbf{k})$
$= 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{DC} = \mathbf{c} - \mathbf{d}$
$= (6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) - (4\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$\overrightarrow{AB} \neq \overrightarrow{DC}$ so $ABCD$ is not a parallelogram

54 $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
$= (-4)(2) + (3)(5) + (1)(-3)$
$= 4$

55 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
$= (3)(8)\cos 45°$
$= 24\left(\frac{\sqrt{2}}{2}\right)$
$= 12\sqrt{2}$

56 $\cos\theta = \frac{a_1b_1+a_2b_2+a_3b_3}{|\mathbf{a}||\mathbf{b}|} = \frac{(1)(4)+(-3)(-2)+(2)(-7)}{\sqrt{1^2+(-3)^2+2^2}\sqrt{4^2+(-2)^2+(-7)^2}}$

$$= \frac{-4}{\sqrt{14}\sqrt{69}}$$

$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{14}\sqrt{69}}\right) = 97.39°$

Required angle is acute so, $180 - 97.39 = 82.6°$

57 $(2\mathbf{a}+3\mathbf{b})\cdot(2\mathbf{a}-3\mathbf{b}) = 2\mathbf{a}\cdot 2\mathbf{a} + 2\mathbf{a}\cdot(-3\mathbf{b}) + 3\mathbf{b}\cdot 2\mathbf{a} + 3\mathbf{b}\cdot(-3\mathbf{b})$

$$\begin{aligned}
&= 4\mathbf{a}\cdot\mathbf{a} - 6\mathbf{a}\cdot\mathbf{b} + 6\mathbf{b}\cdot\mathbf{a} - 9\mathbf{b}\cdot\mathbf{b}\\
&= 4\mathbf{a}\cdot\mathbf{a} - 9\mathbf{b}\cdot\mathbf{b} \qquad \text{(since } \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}\text{)}\\
&= 4|\mathbf{a}|^2 - 9|\mathbf{b}|^2 \qquad \text{(since } \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \text{ and } \mathbf{b}\cdot\mathbf{b} = |\mathbf{b}|^2\text{)}\\
&= 4(25) - 9(4)\\
&= 64
\end{aligned}$$

58 For two vectors to be perpendicular, their scalar product must be zero:

$$\begin{pmatrix} 2+t \\ -3 \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4t-1 \\ 5 \end{pmatrix} = 0$$

$2 + t - 3(4t-1) + 5t = 0$

$1 - 6t = 0$

$t = \frac{1}{6}$

59 Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ with $\mathbf{a} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$:

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

60 A direction vector for the line is given by the vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$:

$$\mathbf{d} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -11 \\ 3 \end{pmatrix}$$

Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$:

$$\mathbf{r} = \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ -11 \\ 3 \end{pmatrix}$$

61 Write $\mathbf{r}$ as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and form a separate equation for each component:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -3+\lambda \\ 4\lambda \\ 5-2\lambda \end{pmatrix}$$

So, $x = -3 + \lambda,\ y = 4\lambda,\ z = 5 - 2\lambda$

62 Set each expression equal to λ and rearrange to express x, y and z in terms of λ:

$$\begin{cases} \frac{x-4}{-3} = \lambda \\ y + 2 = \lambda \\ \frac{z-6}{5} = \lambda \end{cases}$$

$$\begin{cases} x = 4 - 3\lambda \\ y = -2 + \lambda \\ z = 6 + 5\lambda \end{cases}$$

Write as the components of the vector $\mathbf{r}$ and split into a part without λ and a part with λ as a factor:

$$\mathbf{r} = \begin{pmatrix} 4 - 3\lambda \\ -2 + \lambda \\ 6 + 5\lambda \end{pmatrix}$$

So, $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$

63 The angle between the lines is the angle between their direction vectors:

$$\cos\theta = \frac{(2)(1)+(1)(3)+(-1)(-5)}{\sqrt{2^2+1^2+(-1)^2}\sqrt{1^2+3^2+(-5)^2}}$$

$$= \frac{10}{\sqrt{6}\sqrt{35}}$$

$$\theta = \cos^{-1}\left(\frac{10}{\sqrt{6}\sqrt{35}}\right)$$

$$= 46.4°$$

64 a Speed is magnitude of the velocity vector:

$$\text{Speed} = \sqrt{3^2 + (-1)^2 + 2^2}$$

$$= \sqrt{14}\text{ m s}^{-1}$$

b The position vector at time t is given by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$:

$$\mathbf{r} = -5\mathbf{i} + \mathbf{k} + 10(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

$$= 25\mathbf{i} - 10\mathbf{j} + 21\mathbf{k}$$

65 First check whether one direction vector is a multiple of the other:

$$\begin{pmatrix} -4 \\ 10 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

So, the lines have the same direction.

Then check whether the given point on l_1 also lies on l_2 (or vice versa):

$$\begin{pmatrix} 0 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} -4 \\ 10 \\ -6 \end{pmatrix}$$

$$\begin{cases} 0 = 6 - 4\mu & \Rightarrow \mu = 1.5 \\ 8 = -7 + 10\mu & \Rightarrow \mu = 1.5 \\ -5 = 4 - 6\mu & \Rightarrow \mu = 1.5 \end{cases}$$

Since l_1 and l_2 have the same direction and share a common point, they are coincident.

66 Set the position vectors to be equal and try to solve for λ and μ:

$$\begin{pmatrix}5\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}-2\\13\\11\end{pmatrix}+\mu\begin{pmatrix}-2\\3\\4\end{pmatrix}$$

$$\begin{cases}5+\lambda=-2-2\mu & (1)\\ 2-\lambda=13+3\mu & (2)\\ 3+2\lambda=11+4\mu & (3)\end{cases}$$

Solve any pair simultaneously, say (1) and (2):

$$\begin{cases}\lambda+2\mu=-7\\ \lambda+3\mu=-11\end{cases}$$

From GDC: $\lambda=1, \mu=-4$

Check in (3):

$3+2(1)=5$

$11+4(-4)=-5$

So, the lines are skew.

67 Set the position vectors to be equal and solve for λ and μ:

$$\begin{pmatrix}-1\\-7\\5\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\4\end{pmatrix}=\begin{pmatrix}2\\-3\\-6\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$$\begin{cases}-1-3\lambda=2+\mu & (1)\\ -7+\lambda=-3-2\mu & (2)\\ 5+4\lambda=-6+\mu & (3)\end{cases}$$

Solve any pair simultaneously, say (1) and (2):

$$\begin{cases}3\lambda+\mu=-3\\ \lambda+2\mu=4\end{cases}$$

From GDC: $\lambda=-2, \mu=3$

Check in (3):

$5+4(-2)=-3$

$-6+3=-3$

So, the lines intersect.

Substitute $\lambda=-2$ into l_1 (or $\mu=3$ into l_2) to find the point of intersection:

$$\mathbf{r}=\begin{pmatrix}-1\\-7\\5\end{pmatrix}-2\begin{pmatrix}-3\\1\\4\end{pmatrix}=\begin{pmatrix}5\\-9\\-3\end{pmatrix}$$

So, point of intersection is $(5,-9,-3)$

68 Use the result that $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_3b_1 - b_3a_1 \\ a_1b_2 - b_1a_2 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} (4)(-2) - (2)(3) \\ (3)(5) - (-2)(1) \\ (1)(2) - (5)(4) \end{pmatrix}$$
$$= \begin{pmatrix} -14 \\ 17 \\ -18 \end{pmatrix}$$

69 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
$$= (4)(5) \sin 30°$$
$$= 20\left(\frac{1}{2}\right)$$
$$= 10$$

70 $(2\mathbf{a} + \mathbf{b}) \times (5\mathbf{a} + 3\mathbf{b}) = 2\mathbf{a} \times 5\mathbf{a} + 2\mathbf{a} \times 3\mathbf{b} + \mathbf{b} \times 5\mathbf{a} + \mathbf{b} \times 3\mathbf{b}$
$$= 10\mathbf{a} \times \mathbf{a} + 6\mathbf{a} \times \mathbf{b} + 5\mathbf{b} \times \mathbf{a} + 3\mathbf{b} \times \mathbf{b}$$
$$= 0 + 6\mathbf{a} \times \mathbf{b} + 5\mathbf{b} \times \mathbf{a} + 0 \qquad (\text{since } \mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = 0)$$
$$= 6\mathbf{a} \times \mathbf{b} - 5\mathbf{a} \times \mathbf{b} \qquad (\text{since } \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a})$$
$$= \mathbf{a} \times \mathbf{b}$$

71 Any two vectors (starting at the same vertex) are suitable for defining the triangle, for example $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\overrightarrow{PQ} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 7 \end{pmatrix}$$
$$\overrightarrow{PR} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} 5 \\ -5 \\ 7 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ -11 \\ -5 \end{pmatrix}$$
$$\text{Area} = \tfrac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$
$$= \frac{1}{2}\sqrt{(-4)^2 + (-11)^2 + (-5)^2}$$
$$= \frac{9\sqrt{2}}{2}$$

72 Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$ with $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 6 \end{pmatrix}$, $\mathbf{d}_1 = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{d}_2 = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$:

$$\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$$

73 Two vectors parallel to the plane are $\mathbf{d}_1 = \overrightarrow{AB}$ and $\mathbf{d}_2 = \overrightarrow{AC}$:

$$\mathbf{d}_1 = \begin{pmatrix}3\\1\\4\end{pmatrix} - \begin{pmatrix}4\\-6\\0\end{pmatrix} = \begin{pmatrix}-1\\7\\4\end{pmatrix}$$

$$\mathbf{d}_2 = \begin{pmatrix}-2\\2\\1\end{pmatrix} - \begin{pmatrix}4\\-6\\0\end{pmatrix} = \begin{pmatrix}-6\\8\\1\end{pmatrix}$$

Use $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1 + \mu\mathbf{d}_2$:

$$\mathbf{r} = \begin{pmatrix}4\\-6\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\7\\4\end{pmatrix} + \mu\begin{pmatrix}-6\\8\\1\end{pmatrix}$$

Note that the position vector of any of the three points can be used as **a** in the equation of the plane.

74 Use $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ with $\mathbf{n} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix}5\\-8\\4\end{pmatrix}$:

$$\mathbf{r}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix} = \begin{pmatrix}5\\-8\\4\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix} = -15$$

So, $\mathbf{r}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix} = -15$

75 Finding the cross product of the two vectors parallel to the plane will give a vector perpendicular to the plane, i.e. a normal **n**:

$$\mathbf{n} = \begin{pmatrix}2\\2\\1\end{pmatrix} \times \begin{pmatrix}-3\\1\\4\end{pmatrix} = \begin{pmatrix}7\\-11\\8\end{pmatrix}$$

Use $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ with $\mathbf{a} = \begin{pmatrix}9\\2\\-5\end{pmatrix}$:

$$\mathbf{r}\cdot\begin{pmatrix}7\\-11\\8\end{pmatrix} = \begin{pmatrix}9\\2\\-5\end{pmatrix}\cdot\begin{pmatrix}7\\-11\\8\end{pmatrix} = 1$$

So, $\mathbf{r}\cdot\begin{pmatrix}7\\-11\\8\end{pmatrix} = 1$

76 Use the scalar product form but replace **r** with $\begin{pmatrix}x\\y\\z\end{pmatrix}$:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-2\end{pmatrix} = \begin{pmatrix}3\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-2\end{pmatrix}$$

$$4x + y - 2z = 12 + 0 - 10$$
$$4x + y - 2z = 2$$

77 Convert the equation of the line to parametric form:

$x = 1 + 4\lambda$
$y = -3 + \lambda$
$z = -2 - 6\lambda$

Substitute into the equation of the plane and solve for λ:

$2(1 + 4\lambda) - 4(-3 + \lambda) + (-2 - 6\lambda) = 16$
$12 - 2\lambda = 16$
$\lambda = -2$

Use this value of λ in the equation of the line to find the point of intersection:

$$\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} -7 \\ -5 \\ 10 \end{pmatrix}$$

Point of intersection is $(-7, -5, 10)$

78 Eliminate any one of the variables between the two equations:

$$\begin{cases} x + 2y - 3z = 5 & (1) \\ 4x - y + 2z = 11 & (2) \end{cases}$$

$(1) + 2(2)$:

$9x + z = 27$

Let $z = \lambda$:

$9x + \lambda = 27$

$x = 3 - \frac{\lambda}{9}$

Substitute into (1) and rearrange to find an expression for y in terms of λ:

$3 - \frac{\lambda}{9} + 2y - 3\lambda = 5$
$18y = 18 + 28\lambda$
$y = 1 + \frac{14}{9}\lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 - \frac{\lambda}{9} \\ 1 + \frac{14}{9}\lambda \\ \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1/9 \\ 14/9 \\ 1 \end{pmatrix}$$

So, the line of intersection is $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 14 \\ 9 \end{pmatrix}$

79 Solve the system of equation using the GDC:

$x = 3, y = 2, z = -1$

So, the point of intersection is $(3, 2, -1)$

80 Attempt to solve the simultaneous equations. Start by eliminating a variable from any two equations, say z from the first two:

$$\begin{cases} x - 3y + 2z = -7 & (1) \\ 4x + y - z = -5 & (2) \\ 6x - 5y + 3z = 1 & (3) \end{cases}$$

$(4) = (1) + 2(2)$

$(5) = 3(2) + (3)$:

$$\begin{cases} 9x - y = -17 & (4) \\ 18x - 2y = -14 & (5) \\ 6x - 5y + 3z = 1 & (3) \end{cases}$$

Now eliminate y from (4):

$(4) - \frac{1}{2}(5)$:

$0 = -10$

So, the system is inconsistent and therefore the planes don't intersect.

The normals to the three planes are:

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}, \mathbf{n}_3 = \begin{pmatrix} 6 \\ -5 \\ 3 \end{pmatrix}$$

Since none of these normals are parallel to each other (none is a multiple of another), no two planes are parallel.

Therefore, the planes form a triangular prism.

Note that the only other possibilities for non-intersecting planes are that either one plane intersects two parallel planes or the three planes are all parallel.

81 First find a normal to the plane:

$3x - 4y + 2z = 10$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 10$$

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 10$$

So, $\mathbf{n} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$

The angle between a line with direction vector $\mathbf{d}$ and a plane with normal $\mathbf{n}$ is $\theta = 90 - \phi$, where $\cos\phi = \frac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$:

$$\cos\phi = \frac{\left|\begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}\right|}{\sqrt{25+4+1}\sqrt{9+16+4}}$$
$$= \frac{21}{\sqrt{30}\sqrt{29}}$$
$\phi = 44.6°$

$\theta = 90 - 44.6 = 45.4°$

82 First find a normal to each plane:

$$2x + 3y - 5z = 4 \Rightarrow \mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

$$x - 2y + 4z = 9 \Rightarrow \mathbf{n}_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

The angle between two planes is the angle between their normals, so use $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$:

$$\cos\theta = \frac{\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}}{\sqrt{4+9+25}\sqrt{1+4+16}}$$

$$= \frac{-24}{\sqrt{38}\sqrt{21}}$$

$$\theta = 148.2°$$

So, the acute angle between the planes is $180 - 148.2 = 31.8°$

4 Statistics and probability

1 a Discrete – it can only take certain values.

b Continuous (although its measurement might be discrete since it will be to a particular accuracy)

c Discrete

2 a There are many possibilities. It could be all of his patients, all of his ill patients, all people in his area, or all people in the world.

b This is not a random sample of the population since there are some people (those who do not attend the clinic) who cannot possibly be included.

3 Yes – people who do less exercise might be less likely to choose to participate.

4 Yes – there appears to be consistency when the observation is repeated (within a reasonable statistical noise).

5 Item D is not a possible human height. It might have been a participant not taking the test seriously or it might have been a misread (e.g. giving height in feet and inches). You could either return to participant D and ask them to check their response, or if this were not possible you would discard the data item.

6 The IQR is 4. Anything above $11 + 1.5 \times 4 = 17$ is an outlier. Just because a data item is an outlier does not mean that it should be excluded. It should be investigated carefully to ensure that it is still a valid member of the population of interest.

7 Convenience sampling.

8 The proportion from Italy is $\frac{60}{150} = \frac{2}{5}$. The stratified sample must be in the same proportion, so it should contain $\frac{2}{5} \times 20 = 8$ students from Italy.

9 The proportion is $\frac{18+12}{15+18+12} = \frac{30}{45} = \frac{2}{3}$

10 This is all of the 30 to 40 group and half of the 20 to 30 group, which is $7+\frac{4}{2} = 9$

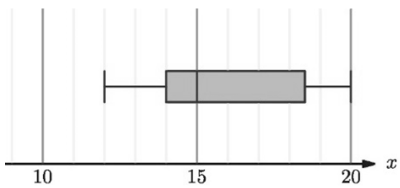
11 a The total frequency is 160. Reading off half of this frequency (80) on the frequency axis is about 42 on the x-axis, which is the median.

b The lower quartile corresponds to a frequency of 40, which is an x-value of approximately 30.

The upper quartile corresponds to a frequency of 120 which is an x-value of approximately 60. Therefore, the interquartile range is $60 - 30 = 30$

c The 90th percentile corresponds to a frequency of $0.9 \times 160 = 144$. This has an x-value of about 72 which is the 90th percentile.

12 Putting the data into the GDC, the following summary statistics can be found:



Min: 12, lower quartile: 14, median: 16, upper quartile: 18.5, max: 20

13 a Both have the similar spread (same IQR (4) and range excluding outliers (10) but population A is higher on average (median of 16 versus 10).

b B is more likely to be normally distributed as it has a symmetric distribution.

14 The mode is 14 as it is the only one which occurs twice. From the GD, the median is 18 and the mean is 18.5

Tip: You should also be able calculate the median without a calculator.

15 The mean is $\frac{23+x}{5} = 7$

So $23 + x = 35$, therefore $x = 12$

16 a The midpoints are 15, 25, 40, 55.

$n = 10 + 12 + 15 + 13 = 50$

$\bar{x} = \frac{10\times15+12\times25+15\times40+13\times55}{50} = \frac{1765}{50} = 35.3$

b We are not using the original data.

17 The modal class is $15 < x \leq 20$

18 a From the GDC, $Q_3 = 18, Q_1 = 7$ so IQR $= 11$

b From the GDC, the standard deviation is 5.45

c The variance is $5.45^2 = 29.7$

19 The new mean is $12 \times 2 + 4 = 28$

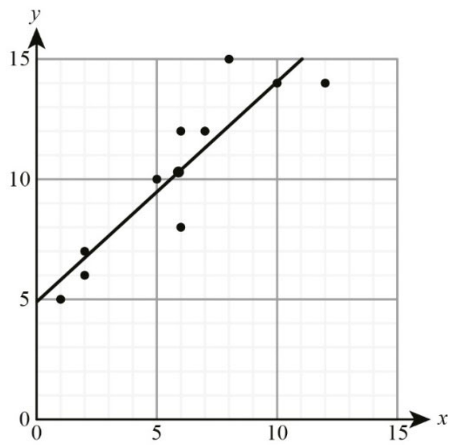
The new standard deviation is $10 \times 2 = 20$.

20 Using the GDC, the lower quartile is 16 and the upper quartile is 28.5

21 a From the GDC: $r = 0.910$

b There is strong positive correlation between x and y

22 a Something like:



b Approximately 7.5

23 From GDC, $y = 0.916x + 4.89$

24 a (i) 13.1

(ii) 23.21

(iii) 5.58

b Only (**i**). Part (**ii**) is extrapolation and part (**iii**) is using a y-on-x line inappropriately.

25 a This is the expected number of text messages sent by a pupil who does not spend any time on social media in a day.

b For every additional hour spent on social media, the model predicts that the pupil will send 1.4 additional texts.

26 a Split the data into the first four points and the next five points and do a regression for each part separately.

$$L = \begin{cases} 4.19A - 0.259 & A < 6 \\ 0.830A + 25.4 & A > 6 \end{cases}$$

b Using the first part of the piecewise graph:

$L = 4.19 \times 3 - 0.259 = 12.3$

So expect a length of 12.3cm

27 $\frac{134}{200} = 0.67$

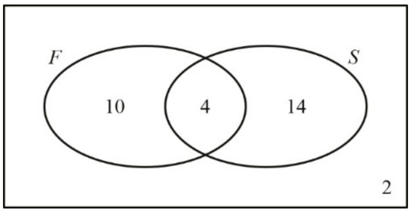
28 There are six possible outcomes of which three (2, 3 and 5) are prime, so the probability is $\frac{3}{6} = 0.5$

29 $\mathrm{P}(A') = 1 - \mathrm{P}(A) = 0.4$

30 $30 \times 0.05 = 1.5$

Tip: Remember that expected values should not be rounded to make them actually achievable.

31 We can illustrate this in a Venn diagram:



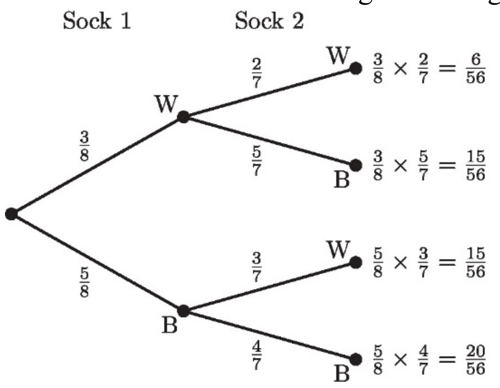
There are $14 - 4 = 10$ students who study only French

There are $18 - 4 = 14$ students who study only Spanish

Therefore, there are $10 + 14 + 4 = 28$ students who study either French or Spanish. This leaves 2 students who do not study either, so the probability is $\frac{2}{30} = \frac{1}{15}$

32 a There will be 7 socks left, of which 5 are black so it is $\frac{5}{7}$

b This is best illustrated using a tree diagram:



There are two branches relevant to the question.

White then black:

$$\frac{3}{8}\times\frac{5}{7}=\frac{15}{56}$$

Black then white:

$$\frac{5}{8}\times\frac{3}{7}=\frac{15}{56}$$

The total probability is $\frac{30}{56}=\frac{15}{28}$

33 a This can be best illustrated using a sample space diagram:

		1st Roll			
		1	**2**	**3**	**4**
2nd Roll	**1**	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

There are 16 places in the sample space diagram: 6 of them have a score above 5 (shaded in the diagram) therefore the probability is $\frac{6}{16}=\frac{3}{8}$

b In the sample space diagram, there are 6 scores above 5. Two of them are 7 so $P(\text{score}=7|\text{score}>5)=\frac{2}{6}=\frac{1}{3}$

34 a $x=100-40-30-20=10$

b There are 60 out of 100 students who prefer Maths, so $\frac{60}{100}=\frac{3}{5}$

35 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.5 + 0.7 - 0.3 = 0.9$

36 $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - 0 = 0.6$

37 There are 70 people who prefer soccer. Out of these 40 prefer maths. So
$\mathrm{P}(\text{Maths}|\text{Soccer}) = \frac{40}{70} = \frac{4}{7}$

38 $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = 0.24$

39 W can take three possible values: 0, 1 or 2

$$\mathrm{P}(W = 0) = \mathrm{P}(BB) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

$$\mathrm{P}(W = 1) = \mathrm{P}(BW) + \mathrm{P}(WB) = \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} = \frac{4}{7}$$

$$\mathrm{P}(W = 2) = \mathrm{P}(WW) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$

So

$\boldsymbol{w}$	0	1	2
$\mathbf{P}(\boldsymbol{W} = \boldsymbol{w})$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

40 Create a table:

$\boldsymbol{x}$	0	1	2
$\mathbf{P}(\boldsymbol{X} = \boldsymbol{x})$	k	$2k$	$3k$

The total probability is $k + 2k + 3k = 6k$ which must equal 1 so $k = \frac{1}{6}$

41 $E(X) = 0.5 \times 0.5 + 1 \times 0.4 + 2.5 \times 0.1 = 0.9$

42 a The probability of winning a prize is $0.095 + 0.005 = 0.1$. Out of this, the probability of winning \$2000 is 0.01, so the conditional probability is $\frac{0.005}{0.1} = 0.05$

b $E(X) = 0 \times 0.9 + 10 \times 0.095 + 2000 \times 0.005 = 10.95$

$\mathrm{P}(X > 10.95) = \mathrm{P}(X = 2000) = 0.005$

43 $E(X) = -1 \times 0.6 + 0 \times 0.3 + 0.1k = 0.1k - 0.6$

If the game is fair then $E(X) = 0$ so:

$0.1k - 0.6 = 0$

$0.1k = 0.6$

$k = 6$

44 The outcome of each trial is not independent of the previous trial.

45 Your calculator should have two functions – one which finds $\mathrm{P}(X = x)$, which we will use in part **a** and one which finds $\mathrm{P}(X \leq x)$ which we will use in part **b**.

a $\mathrm{P}(X = 2) = 0.3456$

b To use the calculator, you need to write the given question into a cumulative probability:

$$\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X \leq 2) = 1 - 0.683 = 0.317$$

46 a If X is the number of heads then $E(X) = np = 10 \times 0.6 = 6$

b $\mathrm{Var}(X) = np(1-p) = 10 \times 0.6 \times 0.4 = 2.4$ so the standard deviation is $\sqrt{2.4} \approx 1.55$

47 We know that about 68% of the data occurs within one standard deviation of the mean, but this takes us to negative values of time which is not possible.

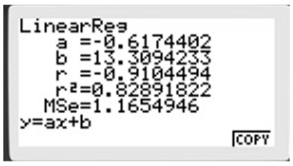
48 a It is a symmetric, bell-shaped curve.

b The line of symmetry is approximately at 50, so this is a good estimate of the mean.

49 $\mathrm{P}(11 < X < 15)$ can be found on the calculator – either directly or as $\mathrm{P}(X < 15) - \mathrm{P}(X < 11)$. It equals 0.625

50 Some calculators can deal with the given information directly, but some require you to first convert the information into a cumulative probability: $\mathrm{P}(X \leq k) = 0.3$. Using the inverse normal function on the calculator gives $k = 92.1$

51 Enter the x-data in the y column and the y-data in the x column. Remember that the output is in the form $x = ay + b$:



Regression line is:

$x = -0.617y + 13.3$

52 a Substitute each value of y into the equation:

(i) $x = 1.82 \times 20 - 11.5 = 24.9$

(ii) $x = 1.82 \times 35 - 11.5 = 52.2$

b The prediction when $y = 20$ can be considered as reliable since 20 is within the range of known y-values and the correlation coefficient is close to 1 suggesting a good linear relationship.

The prediction when $y = 35$ cannot be considered as reliable since the relationship needs to be extrapolated significantly beyond the range of the given data.

53 Use the standard formula with A and B swapped around. Note that $A \cap B$ is the same as $B \cap A$:

$$\begin{aligned}\mathrm{P}(B|A) &= \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}\\ &= \frac{0.4}{0.6}\\ &= \frac{2}{3}\end{aligned}$$

54 A and B are independent if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ so first find $\mathrm{P}(A \cap B)$:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(A) - \mathrm{P}(A \cap B)$$
$$0.7 = 0.2 + 0.8 - \mathrm{P}(A \cap B)$$
$$\mathrm{P}(A \cap B) = 0.3$$

$$\mathrm{P}(A)\mathrm{P}(B) = 0.2 \times 0.8 = 0.16$$

$$\mathrm{P}(A \cap B) \neq \mathrm{P}(A)\mathrm{P}(B)$$

So A and B are not independent

55 The z-value is the number of standard deviations from the mean:

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{17 - 10}{4.8}$$
$$= 1.46$$

17 is 1.46 standard deviations from the mean.

56 Since the mean and variance are unknown, write each probability statement in terms of the standard normal $Z \sim N(0, 1)$ and use the inverse normal to find z_1 and z_2:

$$\mathrm{P}(X < 12) = 0.3$$
$$\mathrm{P}(Z < z_1) = 0.3$$
$$z_1 = -0.52440$$

$$\mathrm{P}(X > 34) = 0.2$$
$$\mathrm{P}(Z > z_2) = 0.2$$
$$z_2 = 0.84162$$

Use $z = \frac{x-\mu}{\sigma}$ to form two equations in μ and σ:

$$-0.52440 = \frac{12 - \mu}{\sigma}$$
$$\mu - 0.52440\sigma = 12 \quad (1)$$

$$0.84162 = \frac{34 - \mu}{\sigma}$$
$$\mu + 0.84162\sigma = 34 \quad (2)$$

Solve (1) and (2) simultaneously on the GDC:

$$\mu = 20.4, \sigma = 16.1$$

57 a Putting all the information into the Bayes' Theorem formula, noting that $P(B') = 0.4$:

$$P(B|A) = \frac{0.6\times0.4}{0.6\times0.4+0.4\times0.8} = \frac{3}{7}$$

b Replacing A by A' in the formula, we now also need $P(A'|B) = 0.6$ and $P(A'|B') = 0.2$.

$$P(B|A') = \frac{0.6\times0.6}{0.6\times0.6+0.4\times0.2} = \frac{9}{11}$$

58 Using a tree diagram may be helpful.

$\text{P(late and Aline)} = 0.2 \times 0.05 = 0.01$

$\text{P(late)} = 0.2 \times 0.05 + 0.35 \times 0.16 + 0.45 \times 0.02 = 0.075$

So $\text{P(Aline | late)} = \dfrac{0.01}{0.075} = 0.133$

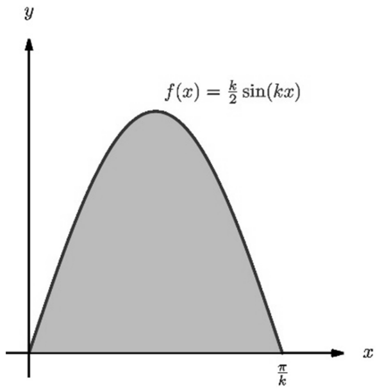
59 First we need $k + 2k + 4k = 1$ so $k = \frac{1}{7}$. Then

$$\text{E}(X) = 0 \times \frac{1}{7} + 1 \times \frac{2}{7} + 2 \times \frac{4}{7} = \frac{10}{7}$$

$$\text{Var}(X) = \left(0 \times \frac{1}{7} + 1 \times \frac{2}{7} + 4 \times \frac{4}{7}\right) - \left(\frac{10}{7}\right)^2 = \frac{26}{49}$$

60 We need $\text{f}(x) \geq 0$ and $\int_0^{\frac{\pi}{k}} \text{f}(x)\text{d}x = 1$.

The former can be seen from the graph:



$$\int_0^{\frac{\pi}{k}} \frac{k}{2}\sin(kx)\,\text{d}x = -\frac{k}{2}\left[\frac{1}{k}\cos(kx)\right]_0^{\frac{\pi}{k}} = -\frac{k}{2k}(\cos\pi - \cos 0] = -\frac{1}{2}(-1-1) = 1$$

61 We first need to find k:

$$\int_0^1 k\sqrt{x}\,\text{d}x = \left[\frac{2k}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{2k}{3} = 1 \;\Rightarrow\; k = \frac{3}{2}$$

Then

$$P\left(X > \frac{1}{4}\right) = \int_{\frac{1}{4}}^1 \frac{3}{2}\sqrt{x}\,\text{d}x = \left[x^{\frac{3}{2}}\right]_{\frac{1}{4}}^1 = 1 - \frac{1}{8} = \frac{7}{8}$$

62 You need to split $\int_{\frac{1}{4}}^{\frac{3}{4}} \text{g}(y)\text{d}y$ into two parts:

$$\text{P}\left(\frac{1}{4} < Y \leq \frac{3}{4}\right) = \frac{4\pi}{4+\pi}\left(\int_{\frac{1}{4}}^{\frac{1}{2}} \sin(\pi y)\,\text{d}y + \int_{\frac{1}{2}}^{\frac{3}{4}} (2-2y)\text{d}y\right) = 0.726 \text{ (3 s.f.)}$$

63 The largest value of the pdf in the given interval is either at a local maximum point or at one of the endpoints. Sketching the graphs shows that the mode of X is $\frac{2}{3}$, but the mode of Y is 0.

64 You can solve the following equation for m using GDC:

$$\int_2^m \frac{\ln x}{\ln 64 - 2}\,\text{d}x = \frac{1}{2}$$

The median is 3.15

65 We first need to find the value of k:

$$\int_0^1 kx\,\mathrm{d}x + \int_1^2 k\,\mathrm{d}x = \frac{k}{2} + k = 1 \quad \Rightarrow \quad k = \frac{2}{3}$$

Now check whether the median is less than 1:

$$\int_0^1 \frac{2}{3}x\,\mathrm{d}x = \frac{1}{3} < \frac{1}{2}$$

So, the median is between 1 and 2, and satisfies:

$$\int_1^m \frac{2}{3}\,\mathrm{d}x = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\Leftrightarrow \left[\frac{2}{3}x\right]_1^m = \frac{2}{3}(m-1) = \frac{1}{6}$$

$$\therefore\ m = \frac{5}{4}$$

66 a $\mathrm{E}(X) = 2 \times \frac{12}{26} + 3 \times \frac{12}{39} + 4 \times \frac{12}{52} = \frac{36}{13}$

$\mathrm{E}(X^2) = 4 \times \frac{12}{26} + 9 \times \frac{12}{39} + 16 \times \frac{12}{52} = \frac{108}{13}$

b $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 0.639$ so $\mathrm{SD}(X) = \sqrt{0.639} = 0.799$

67 $E(X) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x}{2}\cos x\ \mathrm{d}x = 0$

$$\mathrm{E}(X^2) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2}{2}\cos x\ \mathrm{d}x = 0.467$$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = 0.467 - 0^2 = 0.467$

68 $\mathrm{E}(X) = \int_0^2 \frac{x^2}{6}\,\mathrm{d}x + \int_2^6 \left(\frac{x}{2} - \frac{x^2}{12}\right)\mathrm{d}x = \frac{8}{3}$

$$\mathrm{E}(X^2) = \int_0^2 \frac{x^3}{6}\,\mathrm{d}x + \int_2^6 \left(\frac{x^2}{2} - \frac{x^3}{12}\right)\mathrm{d}x = \frac{26}{3}$$

$$\mathrm{Var}\ (X) = \frac{26}{3} - \left(\frac{8}{3}\right)^2 = \frac{14}{9}$$

$$\mathrm{SD}(X) = \sqrt{\frac{14}{9}} = 1.25\ \ (3\text{ s.f.})$$

69 a For a fair game, the cost should equal the expected value of the winnings.

$\mathrm{E}(X) = \int_0^{10} \frac{0.328\,x\,\mathrm{e}^x}{\mathrm{e}^x + \mathrm{e}^7}\,\mathrm{d}x \approx 8$

The charge should be 8 Yen.

b To make a profit, a throw should be longer than 8#m.

$\mathrm{P}(X > 8) = \int_8^{10} \frac{0.328\,\mathrm{e}^x}{\mathrm{e}^x + \mathrm{e}^7}\,\mathrm{d}x \approx 0.569$

Around 57% or players make a profit.

70 $\mathrm{E}(80 - 3X) = 80 - 3\mathrm{E}(X) = 44$

$\mathrm{Var}(X) = (-3)^2 \times 24 = 216$

5 Calculus

1

x	$\frac{\sin(3x)}{0.2x}$
10	0.25
5	0.2588
1	0.2617
0.1	0.2618

The limit is 0.26

2 Look at the values on the graph close to $x = 2$.

The limit is 0.5

3 The derivative is $\frac{\mathrm{d}y}{\mathrm{d}x} = 12 - 5 = 7$.

4 'rate' means $\frac{\mathrm{d}A}{\mathrm{d}t}$; 'decreases' means that the rate of change is negative.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA$$

5 The y value is $\mathrm{f}(x)$ and the gradient is $\mathrm{f}'(x)$. So when $y = 4$, $\mathrm{f}'(x) = -1$.

6 Use $\Delta x = x_Q - 4$, $\Delta y = y_Q - 2$ and gradient $= \frac{\Delta y}{\Delta x}$

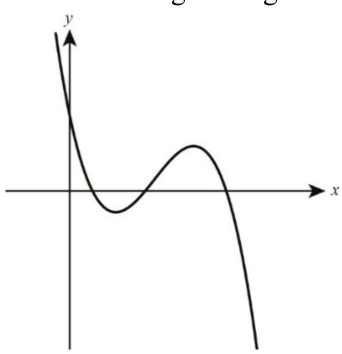
x_Q	y_Q	Δx	Δy	Gradient of PQ
5	2.236	1	0.236	0.236
4.1	2.025	0.1	0.025	0.248
4.01	2.002	0.01	0.002	0.250
4.001	2.000	0.001	0.000	0.250

The gradient is ≈ 0.25

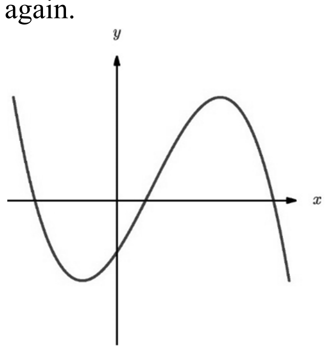
7 $\mathrm{f}'(x)$ is where the graph is decreasing, which is between the two turning points.

$-1.29 < x < 1.29$

8 The gradient starts positive but decreasing, then changes to negative, then back to positive and then to negative again.



9 The gradient starts off negative, so $f(x)$ is decreasing. It then increases, and then decreases again.



10 $\frac{dy}{dx} = 8x + \frac{1}{2}x^{-6} - 3$

11 a $f(x) = 12x^2 - 3x^6$ so $f'(x) = 24x - 18x^5$

b $f(x) = 1 - \frac{3}{2}x^{-4}$, so $f'(x) = \frac{6}{x^5}$ [or $6x^{-5}$]

c $f(x) = \frac{4}{5}x - \frac{3}{5} + \frac{1}{5}x^{-1}$, so $f'(x) = \frac{4}{5} - \frac{1}{5x^2}$ [or $\frac{4}{5} - \frac{1}{5}x^{-2}$]

12 $f'(x) = 8x + 2x^{-2}$

$$f'(2) = 16 + \frac{2}{4} = 16.5$$

13 $\frac{dy}{dx} = 12 - 5x^{-2} = 2$

$$\frac{5}{x^2} = 10, x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

14 $\frac{dy}{dx} = 2x = 8, y = 16 - 3 = 13$

Tangent: $y - 13 = 8(x - 4)$

15 $\frac{dy}{dx} = 3 + 2x^{-2} = 3 + \frac{2}{4} = \frac{7}{2}$

$$y = 6 - \frac{2}{2} = 5$$

Normal: $y - 5 = -\frac{2}{7}(x - 2)$

$$y = -\frac{2}{7}x + \frac{39}{7}$$

16 $\frac{dy}{dx} = 2x$, so the tangent at $(a, a^2 - 3)$ is:

$y - (a^2 - 3) = 2a(x - a)$

When $x = 0, y = -12$:

$-9 - a^2 = -2a^2$

$a = \pm 3$

17 Use GDC to find the gradient and to draw the tangent.

a -0.021

b $y = -0.021x + 0.13$

18 Use GDC to sketch the graph of $\frac{dy}{dx}$ and intersect it with $y = 2$. The coordinates are $(0.5, -0.098)$.

19 $3x^3 - 3x^{-2} + c$

20 $\int \frac{1}{2}x^3 - \frac{3}{2}x^{-2}dx = \frac{1}{8}x^4 + \frac{3}{2}x^{-1} + c$

21 Integrate: $y = \int 4x + 2\,dx = 2x^2 + 2x + c$

Use $y = 3, x = 2$: $3 = 2(2^2) + 2(2) + c \Rightarrow c = -9$

So $y = 2x^2 + 2x - 9$

22 Use GDC: $\int_2^3 2x^3 - 1\,dx = 31.5$

23 $\frac{dy}{dx} = 3\cos x + 5\sin x$

24 $f'(x) = \frac{1}{\sqrt{x}} - \frac{3}{x}$

$$f'(9) = \frac{1}{3} - \frac{3}{9} = 0$$

25 a Using the chain rule with $y = u^{\frac{1}{2}}$ and $u = 3x^2 - 1$:

$$\frac{dy}{dx} = \frac{1}{2}(3x^2 - 1)^{-\frac{1}{2}}(6x) = \frac{3x}{\sqrt{3x^2 - 1}}$$

b Using the chain rule with $y = 2u^3$ and $u = \sin(5x)$, and remembering that $\sin(5x)$ differentiates to $5\cos(5x)$:

$$\frac{dy}{dx} = 6\sin^2(5x) \times 5\cos(5x) = 30\sin^2(5x)\cos(5x)$$

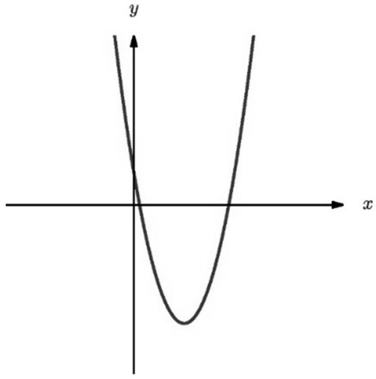
26 $4e^{-3x} - 12xe^{-3x}$

27 Using the quotient rule with $u = \ln x$ and $v = 4x$:

$$\frac{dy}{dx} = \frac{4x\frac{1}{x} - 4\ln x}{16x^2} = \frac{4 - 4\ln x}{16x^2} = \frac{1 - \ln x}{4x^2}$$

28 $6x + \frac{3}{x^2}$

29 The curve is concave-down in the middle section so $f''(x) < 0$ there.



30 Concave-down means that the graph curves downwards, which is at the points B, D and E.

31 Concave up means $f''(x) > 0$. So $f''(x) = 30x - 4 > 0 \therefore x > \frac{2}{15}$

32 $\frac{dy}{dx} = 3x^2 - \frac{24}{x} = 0 \Leftrightarrow 3x^3 = 24 \Leftrightarrow x = 2$

33 $f'(x) = \cos x + \sin x, f'\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$

$$f''(x) = -\sin x + \cos x, f''\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} < 0$$

$$y = f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

34 $S = x^2 + 4 \times \left(x \times \frac{32}{x^2}\right) = x^2 + \frac{128}{x}$

$\frac{dS}{dx} = 2x - \frac{128}{x^2} = 0$ when $x^3 = 64$ so $x = 4$.

$\frac{d^2S}{dx^2} = 2 + \frac{256}{x^3} > 0$, so this is a local minimum.

Min surface area $S = 4^2 + \frac{128}{4} = 48 \text{ cm}^2$

35 $\frac{d^2y}{dx^2} = 60x^3 - 120x^2 = 60x^2(x - 2)$

So $\frac{d^2y}{dx^2} = 0$ when $x = 0$ or 2.

Check for change in sign of $\frac{d^2y}{dx^2}$:

x	-1	1	3
$\frac{d^2y}{dx^2}$	< 0	< 0	> 0

The only point of inflexion is at $x = 2$.

36 $v = \frac{ds}{dt} = 15\cos(5t), a = \frac{dv}{dt} = -75\sin(5t)$

When $t = 2, a = 40.8 \text{ m s}^{-2}$

(or find $\frac{d^2s}{dt^2}$ at $t = 2$ using GDC)

37 Velocity: $v = \frac{\mathrm{d}s}{\mathrm{d}t} = -0.6\mathrm{e}^{-0.2t} = -0.270$

(or find $\frac{\mathrm{d}s}{\mathrm{d}t}$ using GDC)

So speed = 0.270 m s^{-1}

38 $4 + \int_2^5 \frac{1}{\sqrt{t+3}} \mathrm{d}t = 5.18$ m

39 13.8 m

40 $\left(2 \div \frac{1}{3}\right) x^{\frac{1}{3}} + \frac{4}{3}\ln|x| = 6x^{\frac{1}{3}} + \frac{4}{3}\ln|x| + c$

41 $\frac{2\mathrm{e}^{4x}}{4} + \frac{3\mathrm{e}^{-\frac{1}{3}x}}{-\frac{1}{3}} = \frac{1}{2}\mathrm{e}^{4x} - 9\mathrm{e}^{-\frac{1}{3}x} + c$

42 a $\quad 4 \times \frac{\sin^3 x}{3} = \frac{4}{3}\sin^3 x + c$

b $\quad \frac{1}{2}\int\left(\frac{2x}{x^2+3}\right)\mathrm{d}x = \frac{1}{2}\ln(x^2+3) + c$

43 $\left[-\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{6}} = -\frac{1}{2}\left(\frac{1}{2} - 1\right) = \frac{1}{4}$

44 1.79 (3 s.f.)

45 $\int_0^{\frac{\pi}{2}} (\sin x - 1)\,\mathrm{d}x = [-\cos x - x]_0^{\frac{\pi}{2}}$

$\left[0 - \frac{\pi}{2}\right] - [-1 - 0] = 1 - \frac{\pi}{2}$

So area $= \frac{\pi}{2} - 1.$

46 $\mathrm{e}^{-x} = \frac{1}{3}, \mathrm{e}^{x} = 3;\quad x = \ln 3$. The coordinates are (ln 3, 0)

$$\int_0^{\ln 3} (3\mathrm{e}^{-x} - 1)\,\mathrm{d}x = [-3\mathrm{e}^{-x} - x]_0^{\ln 3} = \left[-\frac{3}{\mathrm{e}^{\ln 3}} - \ln 3\right] - [-3\mathrm{e}^0 - 0] = -\frac{3}{3} - \ln 3 + 3$$
$$= 2 - \ln 3$$

$$\int_{\ln 3}^{2} (3\mathrm{e}^{-x} - 1)\,\mathrm{d}x = [-3\mathrm{e}^{-x} - x]_{\ln 3}^{2} = [-3\mathrm{e}^{-2} - 2] - \left[-\frac{3}{3} - \ln 3\right] = -3\mathrm{e}^{-2} - 1 + \ln 3$$

Area = $2 - \ln 3 + 3\mathrm{e}^{-2} + 1 - \ln 3 = 3 - 2\ln 3 + 3\mathrm{e}^{-2}$

47 a Using GDC, points of intersection are $(-4.82, 0.180), (2.69, 7.69)$

b Subtract the bottom curve from the top curve and integrate:

$$A = \int_{-4.82}^{2.69} x + 5 - 2\mathrm{e}^{0.5x}\ \mathrm{d}x = 14.6$$

48 Put $x = 3$ into both expressions; equating those gives $k = -6$

49 *In question, $2x^2$ should be $2x^3$.*

Evaluate $\mathrm{g}(2)$ and $\mathrm{g}'(2)$ for both expressions. They give the same answer, so the function is differentiable at $x = 2$.

50 a Setting $x = 0$ gives $\frac{1}{1-1} = \frac{1}{0}$ which tends to infinity.

b $\frac{4x^2+3}{3x^2-x} = \frac{4+\frac{3}{x^2}}{3-\frac{1}{x}}$; when x tends to infinity, this tends to $\frac{4}{3}$.

51 Simplify $\frac{f(x+h)-f(x)}{h}$ and then take $\lim\limits_{h\to 0}$:

$$\frac{(3(x+h)^2-4)-(3x^2-4)}{h} = \frac{6xh+3h^2}{h}$$
$$= 6x + 3h$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}(6x+3h) = 6x$$

52 Differentiate three times:

$\mathrm{f}'(x) = 2\cos(2x)\,, \mathrm{f}''(x) = -4\sin(2x)\,, \mathrm{f}^{(3)}(x) = -8\cos(2x)$

So $\mathrm{f}^{(3)}\left(\frac{\pi}{6}\right) = -8\cos\left(\frac{\pi}{3}\right) = -4$

53 a Differentiate top and bottom:

$$\frac{e^x}{2\cos(2x)}$$

When $x \to 0$ this tends to $\frac{\mathrm{e}^0}{2\cos(0)} = \frac{1}{2}$.

b Differentiate top and bottom:

$$\frac{\frac{1}{x-2}}{\frac{\pi}{4}\sec^2\left(\frac{\pi x}{4}\right)}$$

When $x \to 2$, top tends to ∞ and bottom tends to $\frac{\pi}{4}\sec^2\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

So, the expression diverges to infinity.

54 The Maclaurin series of $\cos(3x)$ is

$$\cos(3x) \approx 1 - \frac{1}{2}(3x)^2 + \frac{1}{24}(3x)^4 - \cdots = 1 - \frac{9x^2}{2} + \frac{27x^4}{8}$$

So, the expression becomes:

$$\frac{\left(1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \cdots\right) - 1}{2x^2} = -\frac{9}{4} + \frac{27x^2}{16} - \cdots$$

Which tends to $-\frac{9}{4}$ when $x \to 0$

55 It is necessary to use the rules twice:

$$\lim_{x\to\infty}\frac{\mathrm{e}^{0.2x}}{2x^2} = \lim_{x\to\infty}\frac{0.2\mathrm{e}^{0.2x}}{4x}$$
$$= \lim_{x\to\infty}\frac{0.04\mathrm{e}^{0.2x}}{4}$$
$$= \lim_{x\to\infty}0.01\mathrm{e}^{0.2x}$$

The exponential diverges, so the limit is not finite.

56 a Using implicit differentiation:

$$2x - 9y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{9y^2}$$

b Using implicit differentiation and the product rule:

$$\sin y + x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}\cos x + y(-\sin x) = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}(x\cos y + \cos x) = y\sin x - \sin y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\sin x - \sin y}{x\cos y + \cos x}$$

At $\left(0, \frac{\pi}{2}\right)$,

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{0-1}{0+1} = -1$

57 Area of circle decreases at rate of 3 $\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = -3$

And $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

Using the chain rule:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}A} \times \frac{\mathrm{d}A}{\mathrm{d}t}$$
$$= \frac{1}{2\pi r} \times (-3)$$

So, rate of decrease is when $r = 12$ is $\frac{1}{8\pi}$ cm s^{-1}

58 a For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$:

$$3x^2 - 6x = 0$$
$$x = 0 \text{ or } 2$$

Stationary points are $(0, 0)$ and $(2, -4)$

b *In question part b, interval changed from* $-3 \le x \le 3$ *to* $-4 \le x \le 4$

The largest value cannot be attained at $x = -4$ as $(-4)^3 - 3(-4)^2 < 0$ so only need to check at $x = 4$:

$y = 4^3 - 3 \times 4^2 = 16$

Comparing to $y = 0$ and $y = -4$ at the two stationary points, the maximum value is 16

59 a Using the product rule:

$\arctan 2x + x \times \frac{2}{1+(2x)^2} = \arctan 2x + \frac{2x}{1+4x^2}$

b Using the chain rule:

$$\frac{1}{\sec x \ln 2} \times \sec x \tan x = \frac{\tan x}{\ln 2}$$

60 $\int \operatorname{cosec} x\,(\cot x - \operatorname{cosec}^2 x)\,\mathrm{d}x = -\operatorname{cosec} x + \cot x + c$

61 Complete the square inside the square root:

$$\int \frac{2}{\sqrt{1-(2x-1)^2}}\,\mathrm{d}x = 2 \times \frac{1}{2}\arcsin(2x-1)$$

$= \arcsin(2x-1) + c$

62 a $A(x+1) - B(x-2) = 2x + 5$

Substituting in $x = 2$ and $x = -1$ gives $A = 3, B = 1$

b $\int \frac{3}{x-2} - \frac{1}{x+1}\mathrm{d}x = 3\ln|x-2| - \ln|x+1| + c$

Substituting in the limits and sing rules of logs:

$(3\ln 7 - 3\ln 1) - (\ln 10 - \ln 4) = \ln\left(\frac{2\times 7^3}{5}\right)$

63 $x^2 - 4 = 4(\sec^2 u - 1) = 4\tan^2 u$

$\mathrm{d}x = 2\sec u \tan u\ \mathrm{d}u$

So, the integral becomes:

$$\int \frac{1}{2}\mathrm{d}u = \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + c$$

64 $\int_0^4 (u+3)\sqrt{u}\ \mathrm{d}u = \left[\frac{2}{5}u^{\frac{5}{2}} + 3\frac{2}{3}u^{\frac{3}{2}}\right]_0^4$

$= \frac{144}{5}$

65 a Taking $u = x, \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{3x}$ gives $\frac{1}{3}x\mathrm{e}^{3x} - \frac{1}{9}\mathrm{e}^{3x} + c$

b Taking $u = \ln 5x, \frac{\mathrm{d}v}{\mathrm{d}x} = x^4$ gives $\frac{1}{5}x^5\ln 5x - \frac{1}{25}x^5 + c$

66 Take $u = x^2, \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$ to get $\frac{1}{2}x^2\sin 2x - \int x\sin 2x\ \mathrm{d}x$

Then take $u = x, \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x$ to get

$$\frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + c$$

67 $\int_2^6 \mathrm{e}^{\frac{y}{2}}\,\mathrm{d}y = 2(\mathrm{e}^3 - \mathrm{e})$

68 $\int_0^{\pi} \pi\sin^2 x\ \mathrm{d}x = 4.93$ (from GDC)

69 $\int_1^9 \pi\left(\sqrt{y}\right)^2\mathrm{d}y = \left[\pi\frac{y^2}{2}\right]_1^9 = 40\pi$

70 $\frac{d|v|}{dt} = -kv$

71

x	y
0	2.0000
0.1	2.0909
0.2	2.1723
0.3	2.2419
0.4	2.2983

So, $y = 2.298$

72 $\frac{\mathrm{d}y}{\mathrm{d}x} = x(y^2 + 1)$

$$\Rightarrow \int \frac{\mathrm{d}y}{y^2 + 1} = \int x\,\mathrm{d}x$$

$$\Rightarrow \arctan y = \frac{1}{2}x^2 + c$$

$$y = \tan\left(\frac{1}{2}x^2 + c\right)$$

73 $\int \frac{\mathrm{d}y}{y+2} = \int (x-1)\mathrm{d}x$

$$\ln(y + 2) = \frac{1}{2}(x - 1)^2 + c$$

Using $y = 1, x = 1$: $c = \ln 3$

$$y + 2 = \mathrm{e}^{\frac{1}{2}(x-1)^2 + \ln 3}$$

$$y = 3\mathrm{e}^{\frac{1}{2}(x-1)^2} - 2$$

74 a $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + \sqrt{v}$, which simplifies to the required equation.

b Separating variables:

$$\int \frac{\mathrm{d}v}{\sqrt{v}} = \int \frac{\mathrm{d}x}{x} \Rightarrow 2\sqrt{v} = \ln x + c$$

Initial conditions: $x = 1, y = 0, v = 0$, so $c = 0$; gives $y = \frac{x}{4}(\ln x)^2$

75 Integrating factor:

$$I = \mathrm{e}^{\int \frac{2x}{x^2+1}\mathrm{d}x} = \mathrm{e}^{\ln(x^2+1)} = x^2 + 1$$

$$(x^2 + 1)y = \int 4x(x^2 + 1)\mathrm{d}x = x^4 + 2x^2 + c$$

$$\therefore y = \frac{x^4 + 2x^2 + c}{x^2 + 1}$$

76 Let $\mathrm{f}(x) = \ln(1+x)$.

Then:

$$f(0) = 0, f'(0) = \frac{1}{1+0} = 1$$

$$f''(0) = -\frac{1}{(1+0)^2} = -1$$

$$f^3(0) = \frac{2}{(1+0)^3} = 2$$

$$f^{(4)} = -\frac{6}{(1+0)^4} = -6$$

So, $\mathrm{f}(x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

77 $\cos(3x) \approx 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!}$

$$= 1 - \frac{9x^2}{2} + \frac{27x^4}{8}$$

78 $\mathrm{e}^x(1+x)^{-1} \approx \left(1 + x + \frac{x^2}{2}\right)(1 - x + x^2)$

$$= 1 + \frac{3x^2}{2}$$

79 $(1-x^2)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4$

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \approx x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + C$$

$\arcsin(0) = 0$, so

$$\arcsin x \approx x + \frac{x^3}{6} + \frac{3x^5}{40}$$

80 a Write the general Maclaurin series for y and substitute into the differential equation:

$\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k = x + 2y = x + \sum_{k=0}^{\infty} 2a_k x^k$

When $x = 0, y = A$, so $a_0 = A$.

$k = 0$: $a_1 = 2a_0 = 2A$

$k = 1$: $2a_2 = 1 + 2a_1 \Rightarrow a_2 = \frac{1}{2}(1+4A)$

$k \geq 2$: $(k+1)a_{k+1} = 2a_k \Rightarrow a_{k+1} = \frac{2}{k+1}a_k$

b Looking at the first few terms show that $a_k = \frac{2^{k-2}}{k!}(1+4A)$ for all $k \geq 2$. Hence the solution is

$$y = A + 2Ax + \sum_{k=2}^{\infty}\frac{2^{k-2}}{k!}(1+4A)x^k$$

81 a $\frac{d^2y}{dx^2} = -\sin\frac{\pi}{2} = -1$

$\frac{d^3y}{dx^3} = (-\cos y)\frac{dy}{dx} = 0$

$\frac{d^4y}{dx^4} = (\sin y)\frac{dy}{dx} - (\cos y)\frac{d^2y}{dx^2} = 0$

b Hence $y \approx \frac{\pi}{2} - \frac{1}{2}x^2$

Note: The next term is $-\frac{1}{120}x^5$.